1 | FUNCTIONS AND GRAPHS

Figure 1.1 A portion of the San Andreas Fault in California. Major faults like this are the sites of most of the strongest earthquakes ever recorded. (credit: modification of work by Robb Hannawacker, NPS)

Chapter Outline

Introduction

In the past few years, major earthquakes have occurred in several countries around the world. In January 2010, an earthquake of magnitude 7.3 hit Haiti. A magnitude 9 earthquake shook northeastern Japan in March 2011. In April 2014, an 8.2-magnitude earthquake struck off the coast of northern Chile. What do these numbers mean? In particular, how does a magnitude 9 earthquake compare with an earthquake of magnitude 8.2? Or 7.3? Later in this chapter, we show how logarithmic functions are used to compare the relative intensity of two earthquakes based on the magnitude of each earthquake (see **Example 1.39**).

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

1.1 | Review of Functions

Learning Objectives

1.1.1 Use functional notation to evaluate a function.
1.1.2 Determine the domain and range of a function.
1.1.3 Draw the graph of a function.
1.1.4 Find the zeros of a function.
1.1.5 Recognize a function from a table of values.
1.1.6 Make new functions from two or more given functions.
1.1.7 Describe the symmetry properties of a function.

In this section, we provide a formal definition of a function and examine several ways in which functions are represented—namely, through tables, formulas, and graphs. We study formal notation and terms related to functions. We also define composition of functions and symmetry properties. Most of this material will be a review for you, but it serves as a handy reference to remind you of some of the algebraic techniques useful for working with functions.

Functions

Given two sets A and $B,$ a set with elements that are ordered pairs $(x, y),$ where x is an element of A and y is an element of $B,$ is a relation from A to $B.$ A relation from A to B defines a relationship between those two sets. A function is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the *input*; the element of the second set is called the *output*. Functions are used all the time in mathematics to describe relationships between two sets. For any function, when we know the input, the output is determined, so we say that the output is a function of the input. For example, the area of a square is determined by its side length, so we say that the area (the output) is a function of its side length (the input). The velocity of a ball thrown in the air can be described as a function of the amount of time the ball is in the air. The cost of mailing a package is a function of the weight of the package. Since functions have so many uses, it is important to have precise definitions and terminology to study them.

Definition

A **function** f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the **domain** of the function. The set of outputs is called the **range** of the function.

For example, consider the function $f,$ where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9.$ Since every nonnegative real number has a real-value square root, every nonnegative number is an element of the range of this function. Since there is no real number with a square that is negative, the negative real numbers are not elements of the range. We conclude that the range is the set of nonnegative real numbers.

For a general function f with domain $D,$ we often use x to denote the input and y to denote the output associated with $x.$ When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on $x.$ Using function notation, we write $y = f(x),$ and we read this equation as "y equals f of $x.$" For the squaring function described earlier, we write $f(x) = x^2.$

The concept of a function can be visualized using **Figure 1.2**, **Figure 1.3**, and **Figure 1.4**.

Figure 1.2 A function can be visualized as an input/output device.

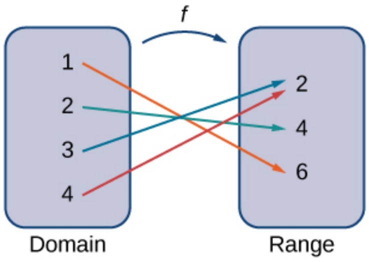


Figure 1.3 A function maps every element in the domain to exactly one element in the range. Although each input can be sent to only one output, two different inputs can be sent to the same output.

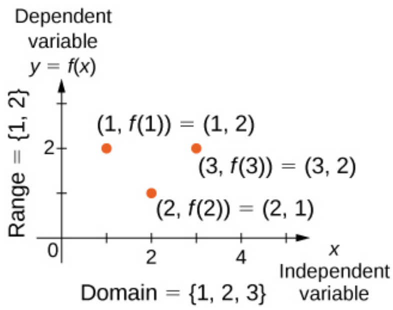


Figure 1.4 In this case, a graph of a function f has a domain of $\{1, 2, 3\}$ and a range of $\{1, 2\}$. The independent variable is x and the dependent variable is y.

Visit this **applet link (http://www.openstax.org/l/grapherrors)** to see more about graphs of functions.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The **graph of a function** is the set of all these points. For example, consider the function f, where the domain is the set $D = \{1, 2, 3\}$ and the rule is $f(x) = 3 - x$. In **Figure 1.5**, we plot a graph of this function.

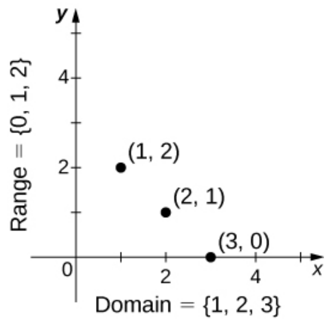


Figure 1.5 Here we see a graph of the function f with domain $\{1, 2, 3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number. For example, since any real number can be squared, if no other domain is specified, we consider the domain of $f(x) = x^2$ to be the set of all real numbers. On the other hand, the square root function $f(x) = \sqrt{x}$ only gives a real output if x is nonnegative. Therefore, the domain of the function $f(x) = \sqrt{x}$ is the set of nonnegative real numbers, sometimes called the *natural domain*.

For the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$, the domains are sets with an infinite number of elements. Clearly we cannot list all these elements. When describing a set with an infinite number of elements, it is often helpful to use set-builder or interval notation. When using set-builder notation to describe a subset of all real numbers, denoted $\mathbb{R}$, we write

$$\{x | x \text{ has some property}\}.$$

We read this as the set of real numbers x such that x has some property. For example, if we were interested in the set of real numbers that are greater than one but less than five, we could denote this set using set-builder notation by writing

$$\{x | 1 < x < 5\}.$$

A set such as this, which contains all numbers greater than a and less than b, can also be denoted using the interval notation (a, b). Therefore,

$$(1, 5) = \{x | 1 < x < 5\}.$$

The numbers 1 and 5 are called the *endpoints* of this set. If we want to consider the set that includes the endpoints, we would denote this set by writing

$$[1, 5] = \{x | 1 \le x \le 5\}.$$

We can use similar notation if we want to include one of the endpoints, but not the other. To denote the set of nonnegative real numbers, we would use the set-builder notation

$$\{x | 0 \le x\}.$$

The smallest number in this set is zero, but this set does not have a largest number. Using interval notation, we would use the symbol ∞, which refers to positive infinity, and we would write the set as

$$[0, \infty) = \{x | 0 \le x\}.$$

It is important to note that ∞ is not a real number. It is used symbolically here to indicate that this set includes all real numbers greater than or equal to zero. Similarly, if we wanted to describe the set of all nonpositive numbers, we could write

$$(-\infty, 0] = \{x | x \le 0\}.$$

Here, the notation $-\infty$ refers to negative infinity, and it indicates that we are including all numbers less than or equal to zero, no matter how small. The set

$$(-\infty, \infty) = \{x | x \text{ is any real number}\}$$

refers to the set of all real numbers.

Some functions are defined using different equations for different parts of their domain. These types of functions are known as *piecewise-defined functions*. For example, suppose we want to define a function f with a domain that is the set of all real numbers such that $f(x) = 3x + 1$ for $x \geq 2$ and $f(x) = x^2$ for $x < 2$. We denote this function by writing

$$f(x) = \begin{cases} 3x + 1 & x \geq 2 \\ x^2 & x < 2 \end{cases}.$$

When evaluating this function for an input x, the equation to use depends on whether $x \geq 2$ or $x < 2$. For example, since $5 > 2$, we use the fact that $f(x) = 3x + 1$ for $x \geq 2$ and see that $f(5) = 3(5) + 1 = 16$. On the other hand, for $x = -1$, we use the fact that $f(x) = x^2$ for $x < 2$ and see that $f(-1) = 1$.

Example 1.1

Evaluating Functions

For the function $f(x) = 3x^2 + 2x - 1$, evaluate

a. $f(-2)$

b. $f(\sqrt{2})$

c. $f(a + h)$

Solution

Substitute the given value for *x* in the formula for $f(x)$.

a. $f(-2) = 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7$

b. $f(\sqrt{2}) = 3(\sqrt{2})^2 + 2\sqrt{2} - 1 = 6 + 2\sqrt{2} - 1 = 5 + 2\sqrt{2}$

c. $$\begin{aligned} f(a + h) = 3(a + h)^2 + 2(a + h) - 1 &= 3\left(a^2 + 2ah + h^2\right) + 2a + 2h - 1 \\ &= 3a^2 + 6ah + 3h^2 + 2a + 2h - 1 \end{aligned}$$

1.1 For $f(x) = x^2 - 3x + 5$, evaluate $f(1)$ and $f(a + h)$.

Example 1.2

Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. $f(x) = (x-4)^2 + 5$

b. $f(x) = \sqrt{3x+2} - 1$

c. $f(x) = \dfrac{3}{x-2}$

Solution

a. Consider $f(x) = (x-4)^2 + 5.$

   i. Since $f(x) = (x-4)^2 + 5$ is a real number for any real number $x,$ the domain of f is the interval $(-\infty, \infty).$

   ii. Since $(x-4)^2 \geq 0,$ we know $f(x) = (x-4)^2 + 5 \geq 5.$ Therefore, the range must be a subset of $\{y|y \geq 5\}.$ To show that every element in this set is in the range, we need to show that for a given y in that set, there is a real number x such that $f(x) = (x-4)^2 + 5 = y.$ Solving this equation for $x,$ we see that we need x such that

$$(x-4)^2 = y - 5.$$

   This equation is satisfied as long as there exists a real number x such that

$$x - 4 = \pm\sqrt{y-5}.$$

   Since $y \geq 5,$ the square root is well-defined. We conclude that for $x = 4 \pm \sqrt{y-5},$ $f(x) = y,$ and therefore the range is $\{y|y \geq 5\}.$

b. Consider $f(x) = \sqrt{3x+2} - 1.$

   i. To find the domain of $f,$ we need the expression $3x + 2 \geq 0.$ Solving this inequality, we conclude that the domain is $\{x|x \geq -2/3\}.$

   ii. To find the range of $f,$ we note that since $\sqrt{3x+2} \geq 0,$ $f(x) = \sqrt{3x+2} - 1 \geq -1.$ Therefore, the range of f must be a subset of the set $\{y|y \geq -1\}.$ To show that every element in this set is in the range of $f,$ we need to show that for all y in this set, there exists a real number x in the domain such that $f(x) = y.$ Let $y \geq -1.$ Then, $f(x) = y$ if and only if

$$\sqrt{3x+2} - 1 = y.$$

   Solving this equation for $x,$ we see that x must solve the equation

$$\sqrt{3x+2} = y + 1.$$

   Since $y \geq -1,$ such an x could exist. Squaring both sides of this equation, we have $3x + 2 = (y+1)^2.$
   Therefore, we need

$$3x = (y+1)^2 - 2,$$

which implies

$$x = \frac{1}{3}(y + 1)^2 - \frac{2}{3}.$$

We just need to verify that x is in the domain of f. Since the domain of f consists of all real numbers greater than or equal to $-2/3$, and

$$\frac{1}{3}(y + 1)^2 - \frac{2}{3} \geq -\frac{2}{3},$$

there does exist an x in the domain of f. We conclude that the range of f is $\{y|y \geq -1\}$.

c. Consider $f(x) = 3/(x - 2)$.

i. Since $3/(x - 2)$ is defined when the denominator is nonzero, the domain is $\{x|x \neq 2\}$.

ii. To find the range of f, we need to find the values of y such that there exists a real number x in the domain with the property that

$$\frac{3}{x - 2} = y.$$

Solving this equation for x, we find that

$$x = \frac{3}{y} + 2.$$

Therefore, as long as $y \neq 0$, there exists a real number x in the domain such that $f(x) = y$. Thus, the range is $\{y|y \neq 0\}$.

1.2 Find the domain and range for $f(x) = \sqrt{4 - 2x} + 5$.

Representing Functions

Typically, a function is represented using one or more of the following tools:

- A table
- A graph
- A formula

We can identify a function in each form, but we can also use them together. For instance, we can plot on a graph the values from a table or create a table from a formula.

Tables

Functions described using a **table of values** arise frequently in real-world applications. Consider the following simple example. We can describe temperature on a given day as a function of time of day. Suppose we record the temperature every hour for a 24-hour period starting at midnight. We let our input variable x be the time after midnight, measured in hours, and the output variable y be the temperature x hours after midnight, measured in degrees Fahrenheit. We record our data in **Table 1.1**.

Hours after Midnight	Temperature $(°F)$	Hours after Midnight	Temperature $(°F)$
0	58	12	84
1	54	13	85
2	53	14	85
3	52	15	83
4	52	16	82
5	55	17	80
6	60	18	77
7	64	19	74
8	72	20	69
9	75	21	65
10	78	22	60
11	80	23	58

Table 1.1 Temperature as a Function of Time of Day

We can see from the table that temperature is a function of time, and the temperature decreases, then increases, and then decreases again. However, we cannot get a clear picture of the behavior of the function without graphing it.

Graphs

Given a function f described by a table, we can provide a visual picture of the function in the form of a graph. Graphing the temperatures listed in **Table 1.1** can give us a better idea of their fluctuation throughout the day. **Figure 1.6** shows the plot of the temperature function.

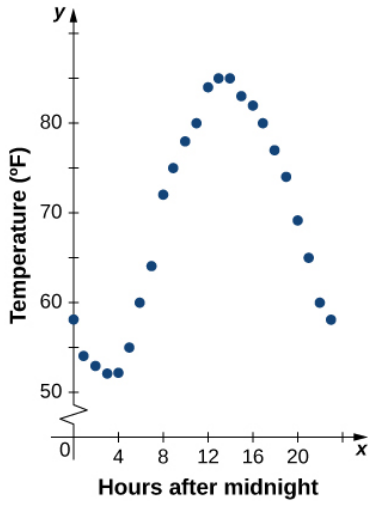


Figure 1.6 The graph of the data from **Table 1.1** shows temperature as a function of time.

From the points plotted on the graph in **Figure 1.6**, we can visualize the general shape of the graph. It is often useful to connect the dots in the graph, which represent the data from the table. In this example, although we cannot make any definitive conclusion regarding what the temperature was at any time for which the temperature was not recorded, given the number of data points collected and the pattern in these points, it is reasonable to suspect that the temperatures at other times followed a similar pattern, as we can see in **Figure 1.7**.

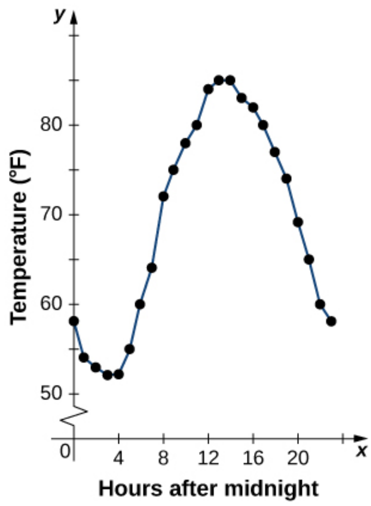


Figure 1.7 Connecting the dots in **Figure 1.6** shows the general pattern of the data.

Algebraic Formulas

Sometimes we are not given the values of a function in table form, rather we are given the values in an explicit formula. Formulas arise in many applications. For example, the area of a circle of radius r is given by the formula $A(r) = \pi r^2$. When an object is thrown upward from the ground with an initial velocity v_0 ft/s, its height above the ground from the time it is thrown until it hits the ground is given by the formula $s(t) = -16t^2 + v_0 t$. When P dollars are invested in an account at an annual interest rate r compounded continuously, the amount of money after t years is given by the formula $A(t) = Pe^{rt}$. Algebraic formulas are important tools to calculate function values. Often we also represent these functions visually in graph form.

Given an algebraic formula for a function f, the graph of f is the set of points $(x, f(x))$, where x is in the domain of f and $f(x)$ is in the range. To graph a function given by a formula, it is helpful to begin by using the formula to create a table of inputs and outputs. If the domain of f consists of an infinite number of values, we cannot list all of them, but because listing some of the inputs and outputs can be very useful, it is often a good way to begin.

When creating a table of inputs and outputs, we typically check to determine whether zero is an output. Those values of x where $f(x) = 0$ are called the **zeros of a function**. For example, the zeros of $f(x) = x^2 - 4$ are $x = \pm 2$. The zeros determine where the graph of f intersects the x-axis, which gives us more information about the shape of the graph of the function. The graph of a function may never intersect the *x*-axis, or it may intersect multiple (or even infinitely many) times.

Another point of interest is the y-intercept, if it exists. The y-intercept is given by $(0, f(0))$.

Since a function has exactly one output for each input, the graph of a function can have, at most, one y-intercept. If $x = 0$ is in the domain of a function f, then f has exactly one y-intercept. If $x = 0$ is not in the domain of f, then f has no y-intercept. Similarly, for any real number c, if c is in the domain of f, there is exactly one output $f(c)$, and the line $x = c$ intersects the graph of f exactly once. On the other hand, if c is not in the domain of f, $f(c)$ is not defined and the line $x = c$ does not intersect the graph of f. This property is summarized in the **vertical line test**.

Rule: Vertical Line Test

Given a function f, every vertical line that may be drawn intersects the graph of f no more than once. If any vertical line intersects a set of points more than once, the set of points does not represent a function.

We can use this test to determine whether a set of plotted points represents the graph of a function (**Figure 1.8**).

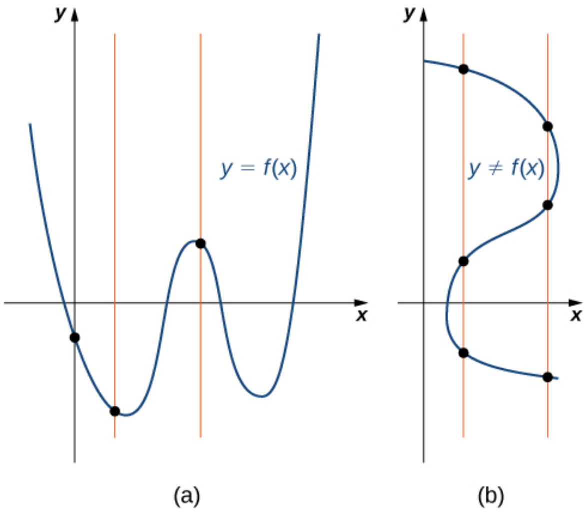


Figure 1.8 (a) The set of plotted points represents the graph of a function because every vertical line intersects the set of points, at most, once. (b) The set of plotted points does not represent the graph of a function because some vertical lines intersect the set of points more than once.

Example 1.3

Finding Zeros and y-Intercepts of a Function

Consider the function $f(x) = -4x + 2$.

a. Find all zeros of f.

b. Find the y-intercept (if any).

c. Sketch a graph of f.

Solution

a. To find the zeros, solve $f(x) = -4x + 2 = 0$. We discover that f has one zero at $x = 1/2$.

b. The y-intercept is given by $(0, f(0)) = (0, 2)$.

c. Given that f is a linear function of the form $f(x) = mx + b$ that passes through the points $(1/2, 0)$ and $(0, 2)$, we can sketch the graph of f (**Figure 1.9**).

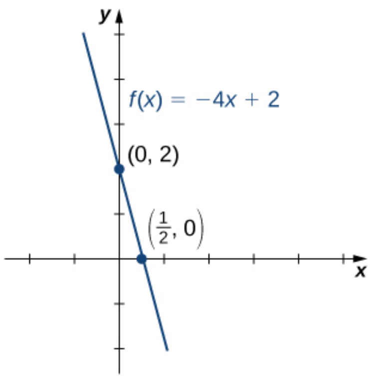


Figure 1.9 The function $f(x) = -4x + 2$ is a line with x-intercept $(1/2, 0)$ and y-intercept $(0, 2)$.

Example 1.4

Using Zeros and y-Intercepts to Sketch a Graph

Consider the function $f(x) = \sqrt{x+3} + 1$.

a. Find all zeros of f.

b. Find the y-intercept (if any).

c. Sketch a graph of f.

Solution

a. To find the zeros, solve $\sqrt{x+3} + 1 = 0$. This equation implies $\sqrt{x+3} = -1$. Since $\sqrt{x+3} \geq 0$ for all

$x,$ this equation has no solutions, and therefore f has no zeros.

b. The y-intercept is given by $(0, f(0)) = (0, \sqrt{3} + 1).$

c. To graph this function, we make a table of values. Since we need $x + 3 \geq 0,$ we need to choose values of $x \geq -3.$ We choose values that make the square-root function easy to evaluate.

x	-3	-2	1
$f(x)$	1	2	3

Table 1.2

Making use of the table and knowing that, since the function is a square root, the graph of f should be similar to the graph of $y = \sqrt{x},$ we sketch the graph (**Figure 1.10**).

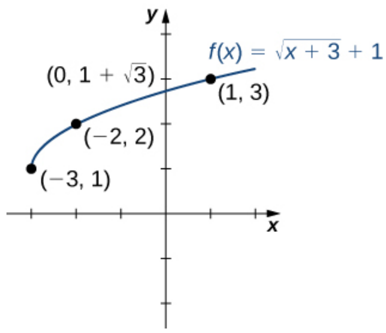


Figure 1.10 The graph of $f(x) = \sqrt{x + 3} + 1$ has a y-intercept but no x-intercepts.

1.3 Find the zeros of $f(x) = x^3 - 5x^2 + 6x.$

Example 1.5

Finding the Height of a Free-Falling Object

If a ball is dropped from a height of 100 ft, its height s at time t is given by the function $s(t) = -16t^2 + 100,$ where s is measured in feet and t is measured in seconds. The domain is restricted to the interval $[0, c],$ where $t = 0$ is the time when the ball is dropped and $t = c$ is the time when the ball hits the ground.

a. Create a table showing the height $s(t)$ when $t = 0, 0.5, 1, 1.5, 2,$ and $2.5.$ Using the data from the table, determine the domain for this function. That is, find the time c when the ball hits the ground.

b. Sketch a graph of $s.$

Solution

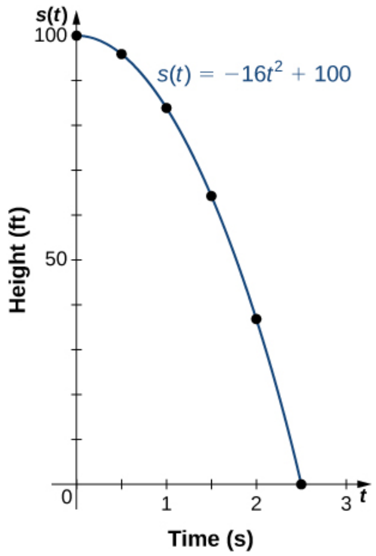
a.

t	0	0.5	1	1.5	2	2.5
$s(t)$	100	96	84	64	36	0

Table 1.3
Height s as a Function of Time t

Since the ball hits the ground when $t = 2.5$, the domain of this function is the interval $[0, 2.5]$.

b.



Note that for this function and the function $f(x) = -4x + 2$ graphed in **Figure 1.9**, the values of $f(x)$ are getting smaller as x is getting larger. A function with this property is said to be decreasing. On the other hand, for the function $f(x) = \sqrt{x+3} + 1$ graphed in **Figure 1.10**, the values of $f(x)$ are getting larger as the values of x are getting larger. A function with this property is said to be increasing. It is important to note, however, that a function can be increasing on some interval or intervals and decreasing over a different interval or intervals. For example, using our temperature function in **Figure 1.6**, we can see that the function is decreasing on the interval $(0, 4)$, increasing on the interval $(4, 14)$, and then decreasing on the interval $(14, 23)$. We make the idea of a function increasing or decreasing over a particular interval more precise in the next definition.

Definition

We say that a function f is **increasing on the interval I** if for all $x_1, x_2 \in I$,

$$f(x_1) \le f(x_2) \text{ when } x_1 < x_2.$$

We say f is strictly increasing on the interval I if for all $x_1, x_2 \in I$,

$$f(x_1) < f(x_2) \text{ when } x_1 < x_2.$$

We say that a function f is **decreasing on the interval I** if for all $x_1, x_2 \in I,$

$$f(x_1) \geq f(x_2) \text{ if } x_1 < x_2.$$

We say that a function f is strictly decreasing on the interval I if for all $x_1, x_2 \in I,$

$$f(x_1) > f(x_2) \text{ if } x_1 < x_2.$$

For example, the function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$ because $3x_1 < 3x_2$ whenever $x_1 < x_2$. On the other hand, the function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$ because $-x_1^3 > -x_2^3$ whenever $x_1 < x_2$ (**Figure 1.11**).

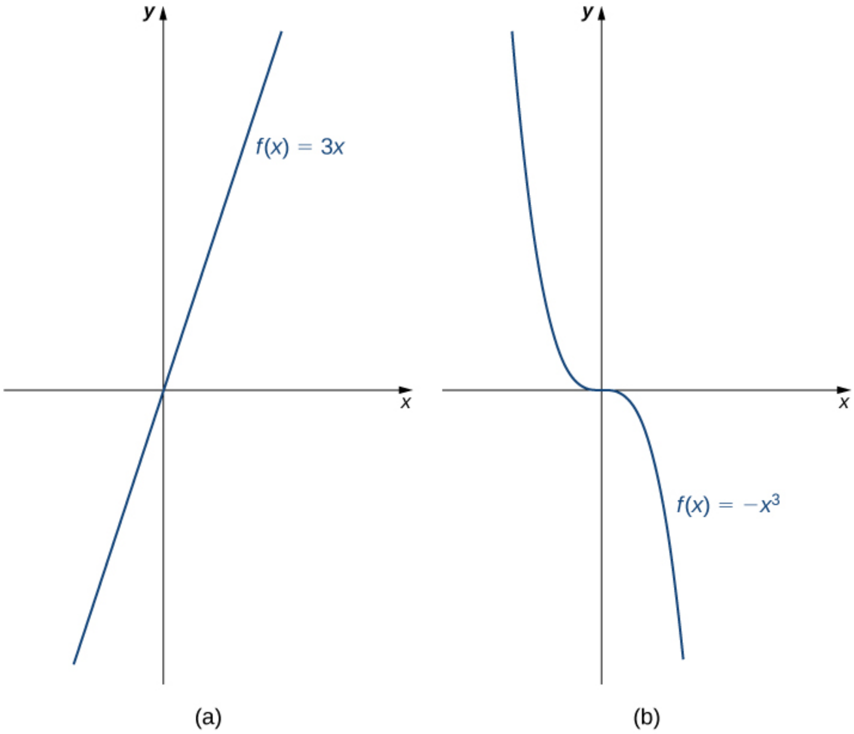


Figure 1.11 (a) The function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$. (b) The function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$.

Combining Functions

Now that we have reviewed the basic characteristics of functions, we can see what happens to these properties when we combine functions in different ways, using basic mathematical operations to create new functions. For example, if the cost for a company to manufacture x items is described by the function $C(x)$ and the revenue created by the sale of x items is described by the function $R(x)$, then the profit on the manufacture and sale of x items is defined as $P(x) = R(x) - C(x)$. Using the difference between two functions, we created a new function.

Alternatively, we can create a new function by composing two functions. For example, given the functions $f(x) = x^2$ and $g(x) = 3x + 1$, the composite function $f \circ g$ is defined such that

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (3x + 1)^2.$$

The composite function $g \circ f$ is defined such that

$$(g \circ f)(x) = g(f(x)) = 3f(x) + 1 = 3x^2 + 1.$$

Note that these two new functions are different from each other.

Combining Functions with Mathematical Operators

To combine functions using mathematical operators, we simply write the functions with the operator and simplify. Given two functions f and g, we can define four new functions:

$$\begin{array}{ll} (f+g)(x) = f(x) + g(x) & \textit{Sum} \\ (f-g)(x) = f(x) - g(x) & \textit{Difference} \\ (f \cdot g)(x) = f(x)g(x) & \textit{Product} \\ \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ for } g(x) \neq 0 & \textit{Quotient} \end{array}$$

Example 1.6

Combining Functions Using Mathematical Operations

Given the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, find each of the following functions and state its domain.

a. $(f+g)(x)$

b. $(f-g)(x)$

c. $(f \cdot g)(x)$

d. $\left(\frac{f}{g}\right)(x)$

Solution

a. $(f+g)(x) = (2x-3) + (x^2-1) = x^2 + 2x - 4.$ The domain of this function is the interval $(-\infty, \infty)$.

b. $(f-g)(x) = (2x-3) - (x^2-1) = -x^2 + 2x - 2.$ The domain of this function is the interval $(-\infty, \infty)$.

c. $(f \cdot g)(x) = (2x-3)(x^2-1) = 2x^3 - 3x^2 - 2x + 3.$ The domain of this function is the interval $(-\infty, \infty)$.

d. $\left(\frac{f}{g}\right)(x) = \frac{2x-3}{x^2-1}.$ The domain of this function is $\{x | x \neq \pm 1\}$.

1.4 For $f(x) = x^2 + 3$ and $g(x) = 2x - 5$, find $(f/g)(x)$ and state its domain.

Function Composition

When we compose functions, we take a function of a function. For example, suppose the temperature T on a given day is described as a function of time t (measured in hours after midnight) as in **Table 1.1**. Suppose the cost C, to heat or cool a building for 1 hour, can be described as a function of the temperature T. Combining these two functions, we can describe the cost of heating or cooling a building as a function of time by evaluating $C(T(t))$. We have defined a new function, denoted $C \circ T$, which is defined such that $(C \circ T)(t) = C(T(t))$ for all t in the domain of T. This new function is called a composite function. We note that since cost is a function of temperature and temperature is a function of time, it makes sense to define this new function $(C \circ T)(t)$. It does not make sense to consider $(T \circ C)(t)$, because temperature is not a function of cost.

Definition

Consider the function f with domain A and range B, and the function g with domain D and range E. If B is a subset of D, then the **composite function** $(g \circ f)(x)$ is the function with domain A such that

$$(g \circ f)(x) = g(f(x)). \tag{1.1}$$

A composite function $g \circ f$ can be viewed in two steps. First, the function f maps each input x in the domain of f to its output $f(x)$ in the range of f. Second, since the range of f is a subset of the domain of g, the output $f(x)$ is an element in the domain of g, and therefore it is mapped to an output $g(f(x))$ in the range of g. In **Figure 1.12**, we see a visual image of a composite function.

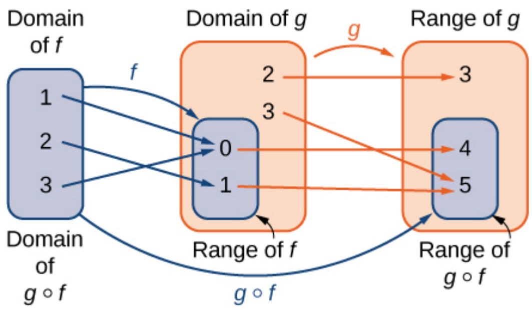


Figure 1.12 For the composite function $g \circ f$, we have $(g \circ f)(1) = 4$, $(g \circ f)(2) = 5$, and $(g \circ f)(3) = 4$.

Example 1.7

Compositions of Functions Defined by Formulas

Consider the functions $f(x) = x^2 + 1$ and $g(x) = 1/x$.

a. Find $(g \circ f)(x)$ and state its domain and range.

b. Evaluate $(g \circ f)(4)$, $(g \circ f)(-1/2)$.

c. Find $(f \circ g)(x)$ and state its domain and range.

d. Evaluate $(f \circ g)(4)$, $(f \circ g)(-1/2)$.

Solution

a. We can find the formula for $(g \circ f)(x)$ in two different ways. We could write

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = \frac{1}{x^2 + 1}.$$

Alternatively, we could write

$$(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{x^2 + 1}.$$

Since $x^2 + 1 \neq 0$ for all real numbers x, the domain of $(g \circ f)(x)$ is the set of all real numbers. Since $0 < 1/(x^2 + 1) \leq 1$, the range is, at most, the interval $(0, 1]$. To show that the range is this entire interval, we let $y = 1/(x^2 + 1)$ and solve this equation for x to show that for all y in the interval $(0, 1]$, there exists a real number x such that $y = 1/(x^2 + 1)$. Solving this equation for x, we see that $x^2 + 1 = 1/y$, which implies that

$$x = \pm\sqrt{\frac{1}{y} - 1}.$$

If y is in the interval $(0, 1]$, the expression under the radical is nonnegative, and therefore there exists a real number x such that $1/(x^2 + 1) = y$. We conclude that the range of $g \circ f$ is the interval $(0, 1]$.

b. $(g \circ f)(4) = g(f(4)) = g(4^2 + 1) = g(17) = \frac{1}{17}$

$(g \circ f)\left(-\frac{1}{2}\right) = g\left(f\left(-\frac{1}{2}\right)\right) = g\left(\left(-\frac{1}{2}\right)^2 + 1\right) = g\left(\frac{5}{4}\right) = \frac{4}{5}$

c. We can find a formula for $(f \circ g)(x)$ in two ways. First, we could write

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^2 + 1.$$

Alternatively, we could write

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 + 1 = \left(\frac{1}{x}\right)^2 + 1.$$

The domain of $f \circ g$ is the set of all real numbers x such that $x \neq 0$. To find the range of f, we need to find all values y for which there exists a real number $x \neq 0$ such that

$$\left(\frac{1}{x}\right)^2 + 1 = y.$$

Solving this equation for x, we see that we need x to satisfy

$$\left(\frac{1}{x}\right)^2 = y - 1,$$

which simplifies to

$$\frac{1}{x} = \pm\sqrt{y - 1}.$$

Finally, we obtain

$$x = \pm \frac{1}{\sqrt{y-1}}.$$

Since $1/\sqrt{y-1}$ is a real number if and only if $y > 1$, the range of f is the set $\{y | y > 1\}$.

d. $(f \circ g)(4) = f(g(4)) = f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 + 1 = \frac{17}{16}$

$(f \circ g)\left(-\frac{1}{2}\right) = f\left(g\left(-\frac{1}{2}\right)\right) = f(-2) = (-2)^2 + 1 = 5$

In **Example 1.7**, we can see that $(f \circ g)(x) \neq (g \circ f)(x)$. This tells us, in general terms, that the order in which we compose functions matters.

1.5 Let $f(x) = 2 - 5x$. Let $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$.

Example 1.8

Composition of Functions Defined by Tables

Consider the functions f and g described by **Table 1.4** and **Table 1.5**.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	4	2	4	-2	0	-2	4

Table 1.4

x	-4	-2	0	2	4
$g(x)$	1	0	3	0	5

Table 1.5

a. Evaluate $(g \circ f)(3)$, $(g \circ f)(0)$.

b. State the domain and range of $(g \circ f)(x)$.

c. Evaluate $(f \circ f)(3)$, $(f \circ f)(1)$.

d. State the domain and range of $(f \circ f)(x)$.

Solution

a. $(g \circ f)(3) = g(f(3)) = g(-2) = 0$
$(g \circ f)(0) = g(4) = 5$

b. The domain of $g \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $g \circ f$ is the set $\{0, 3, 5\}$.

c. $(f \circ f)(3) = f(f(3)) = f(-2) = 4$
$(f \circ f)(1) = f(f(1)) = f(-2) = 4$

d. The domain of $f \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $f \circ f$ is the set $\{0, 4\}$.

Example 1.9

Application Involving a Composite Function

A store is advertising a sale of 20% off all merchandise. Caroline has a coupon that entitles her to an additional 15% off any item, including sale merchandise. If Caroline decides to purchase an item with an original price of x dollars, how much will she end up paying if she applies her coupon to the sale price? Solve this problem by using a composite function.

Solution

Since the sale price is 20% off the original price, if an item is x dollars, its sale price is given by $f(x) = 0.80x$. Since the coupon entitles an individual to 15% off the price of any item, if an item is y dollars, the price, after applying the coupon, is given by $g(y) = 0.85y$. Therefore, if the price is originally x dollars, its sale price will be $f(x) = 0.80x$ and then its final price after the coupon will be $g(f(x)) = 0.85(0.80x) = 0.68x$.

1.6 If items are on sale for 10% off their original price, and a customer has a coupon for an additional 30% off, what will be the final price for an item that is originally x dollars, after applying the coupon to the sale price?

Symmetry of Functions

The graphs of certain functions have symmetry properties that help us understand the function and the shape of its graph. For example, consider the function $f(x) = x^4 - 2x^2 - 3$ shown in **Figure 1.13**(a). If we take the part of the curve that lies to the right of the y-axis and flip it over the y-axis, it lays exactly on top of the curve to the left of the y-axis. In this case, we say the function has **symmetry about the y-axis**. On the other hand, consider the function $f(x) = x^3 - 4x$ shown in **Figure 1.13**(b). If we take the graph and rotate it $180°$ about the origin, the new graph will look exactly the same. In this case, we say the function has **symmetry about the origin**.

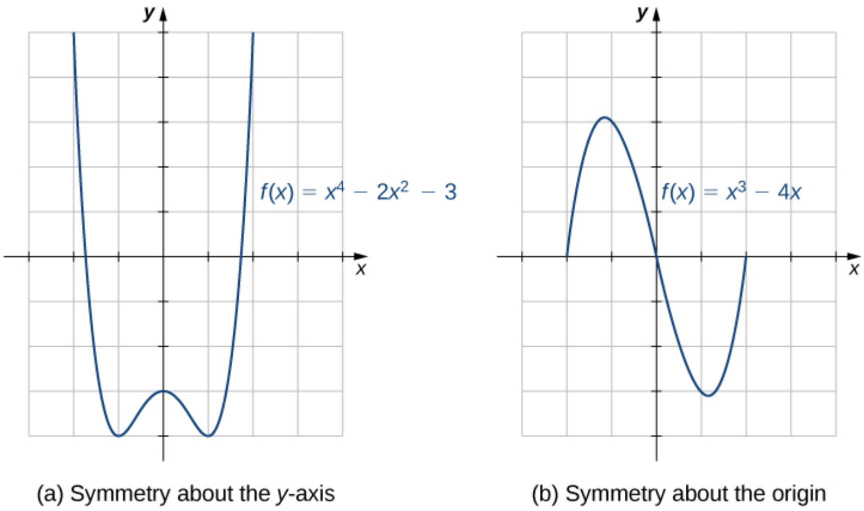


Figure 1.13 (a) A graph that is symmetric about the y-axis. (b) A graph that is symmetric about the origin.

If we are given the graph of a function, it is easy to see whether the graph has one of these symmetry properties. But without a graph, how can we determine algebraically whether a function f has symmetry? Looking at **Figure 1.14** again, we see that since f is symmetric about the y-axis, if the point (x, y) is on the graph, the point $(-x, y)$ is on the graph. In other words, $f(-x) = f(x)$. If a function f has this property, we say f is an even function, which has symmetry about the y-axis. For example, $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

In contrast, looking at **Figure 1.14** again, if a function f is symmetric about the origin, then whenever the point (x, y) is on the graph, the point $(-x, -y)$ is also on the graph. In other words, $f(-x) = -f(x)$. If f has this property, we say f is an odd function, which has symmetry about the origin. For example, $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Definition

If $f(x) = f(-x)$ for all x in the domain of f, then f is an **even function**. An even function is symmetric about the y-axis.

If $f(-x) = -f(x)$ for all x in the domain of f, then f is an **odd function**. An odd function is symmetric about the origin.

Example 1.10

Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither.

a. $f(x) = -5x^4 + 7x^2 - 2$

b. $f(x) = 2x^5 - 4x + 5$

c. $f(x) = \dfrac{3x}{x^2 + 1}$

Solution

To determine whether a function is even or odd, we evaluate $f(-x)$ and compare it to $f(x)$ and $-f(x)$.

a. $f(-x) = -5(-x)^4 + 7(-x)^2 - 2 = -5x^4 + 7x^2 - 2 = f(x)$. Therefore, f is even.

b. $f(-x) = 2(-x)^5 - 4(-x) + 5 = -2x^5 + 4x + 5$. Now, $f(-x) \neq f(x)$. Furthermore, noting that $-f(x) = -2x^5 + 4x - 5$, we see that $f(-x) \neq -f(x)$. Therefore, f is neither even nor odd.

c. $f(-x) = 3(-x)/((-x)^2 + 1\} = -3x/(x^2 + 1) = -[3x/(x^2 + 1)] = -f(x)$. Therefore, f is odd.

1.7 Determine whether $f(x) = 4x^3 - 5x$ is even, odd, or neither.

One symmetric function that arises frequently is the **absolute value function**, written as $|x|$. The absolute value function is defined as

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}. \tag{1.2}$$

Some students describe this function by stating that it "makes everything positive." By the definition of the absolute value function, we see that if $x < 0$, then $|x| = -x > 0$, and if $x > 0$, then $|x| = x > 0$. However, for $x = 0$, $|x| = 0$. Therefore, it is more accurate to say that for all nonzero inputs, the output is positive, but if $x = 0$, the output $|x| = 0$. We conclude that the range of the absolute value function is $\{y | y \geq 0\}$. In **Figure 1.14**, we see that the absolute value function is symmetric about the y-axis and is therefore an even function.

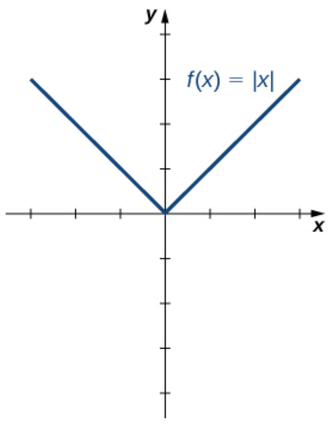


Figure 1.14 The graph of $f(x) = |x|$ is symmetric about the y-axis.

Example 1.11

Working with the Absolute Value Function

Find the domain and range of the function $f(x) = 2|x - 3| + 4$.

Solution

Since the absolute value function is defined for all real numbers, the domain of this function is $(-\infty, \infty)$. Since $|x - 3| \geq 0$ for all x, the function $f(x) = 2|x - 3| + 4 \geq 4$. Therefore, the range is, at most, the set $\{y|y \geq 4\}$. To see that the range is, in fact, this whole set, we need to show that for $y \geq 4$ there exists a real number x such that

$$2|x - 3| + 4 = y.$$

A real number x satisfies this equation as long as

$$|x - 3| = \frac{1}{2}(y - 4).$$

Since $y \geq 4$, we know $y - 4 \geq 0$, and thus the right-hand side of the equation is nonnegative, so it is possible that there is a solution. Furthermore,

$$|x - 3| = \begin{cases} -(x - 3) & \text{if } x < 3 \\ x - 3 & \text{if } x \geq 3 \end{cases}.$$

Therefore, we see there are two solutions:

$$x = \pm\frac{1}{2}(y - 4) + 3.$$

The range of this function is $\{y|y \geq 4\}$.

1.8 For the function $f(x) = |x + 2| - 4$, find the domain and range.

1.1 EXERCISES

For the following exercises, (a) determine the domain and the range of each relation, and (b) state whether the relation is a function.

1.

x	y	x	y
−3	9	1	1
−2	4	2	4
−1	1	3	9
0	0		

2.

x	y	x	y
−3	−2	1	1
−2	−8	2	8
−1	−1	3	−2
0	0		

3.

x	y	x	y
1	−3	1	1
2	−2	2	2
3	−1	3	3
0	0		

4.

x	y	x	y
1	1	5	1
2	1	6	1
3	1	7	1
4	1		

5.

x	y	x	y
3	3	15	1
5	2	21	2
8	1	33	3
10	0		

6.

x	y	x	y
−7	11	1	−2
−2	5	3	4
−2	1	6	11
0	−1		

For the following exercises, find the values for each function, if they exist, then simplify.

a. $f(0)$ b. $f(1)$ c. $f(3)$ d. $f(-x)$ e. $f(a)$ f. $f(a+h)$

7. $f(x) = 5x - 2$

8. $f(x) = 4x^2 - 3x + 1$

9. $f(x) = \frac{2}{x}$

10. $f(x) = |x - 7| + 8$

11. $f(x) = \sqrt{6x + 5}$

12. $f(x) = \frac{x - 2}{3x + 7}$

13. $f(x) = 9$

For the following exercises, find the domain, range, and all zeros/intercepts, if any, of the functions.

14. $f(x) = \frac{x}{x^2 - 16}$

15. $g(x) = \sqrt{8x - 1}$

16. $h(x) = \frac{3}{x^2 + 4}$

17. $f(x) = -1 + \sqrt{x + 2}$

18. $f(x) = \frac{1}{\sqrt{x - 9}}$

19. $g(x) = \frac{3}{x - 4}$

20. $f(x) = 4|x + 5|$

21. $g(x) = \sqrt{\frac{7}{x - 5}}$

For the following exercises, set up a table to sketch the graph of each function using the following values: $x = -3, -2, -1, 0, 1, 2, 3.$

22. $f(x) = x^2 + 1$

x	***y***	***x***	***y***
−3	10	1	2
−2	5	2	5
−1	2	3	10
0	1		

23. $f(x) = 3x - 6$

x	***y***	***x***	***y***
−3	−15	1	−3
−2	−12	2	0
−1	−9	3	3
0	−6		

24. $f(x) = \frac{1}{2}x + 1$

x	***y***	***x***	***y***
−3	$-\frac{1}{2}$	1	$\frac{3}{2}$
−2	0	2	2
−1	$\frac{1}{2}$	3	$\frac{5}{2}$
0	1		

25. $f(x) = 2|x|$

x	***y***	***x***	***y***
−3	6	1	2
−2	4	2	4
−1	2	3	6
0	0		

26. $f(x) = -x^2$

x	y	x	y
−3	−9	1	−1
−2	−4	2	−4
−1	−1	3	−9
0	0		

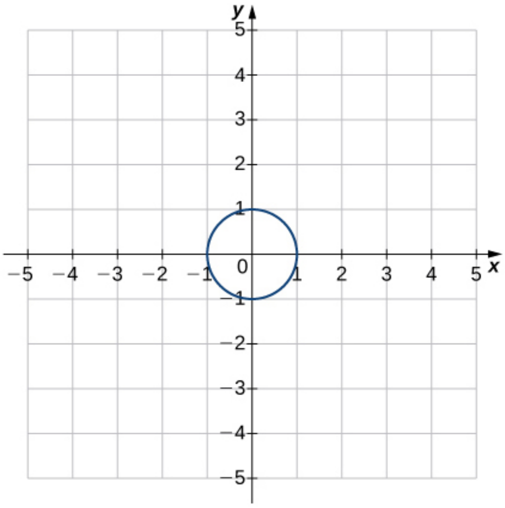
27. $f(x) = x^3$

x	y	x	y
−3	−27	1	1
−2	−8	2	8
−1	−1	3	27
0	0		

For the following exercises, use the vertical line test to determine whether each of the given graphs represents a function. **Assume that a graph continues at both ends if it extends beyond the given grid.** If the graph represents a function, then determine the following for each graph:

a. Domain and range
b. x-intercept, if any (estimate where necessary)
c. y-Intercept, if any (estimate where necessary)
d. The intervals for which the function is increasing
e. The intervals for which the function is decreasing
f. The intervals for which the function is constant
g. Symmetry about any axis and/or the origin
h. Whether the function is even, odd, or neither

28.



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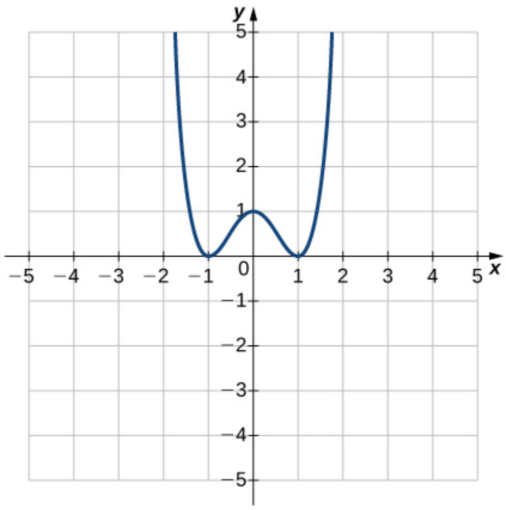

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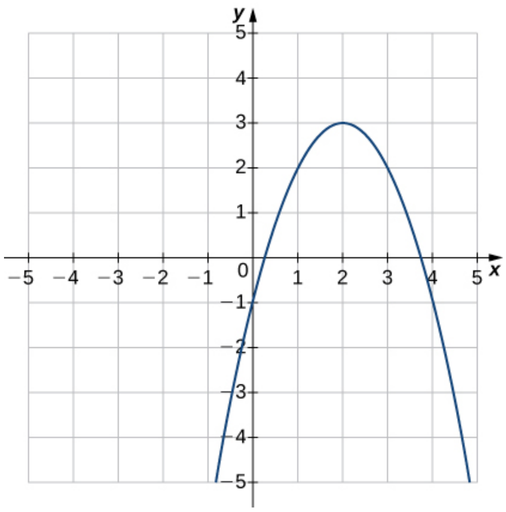

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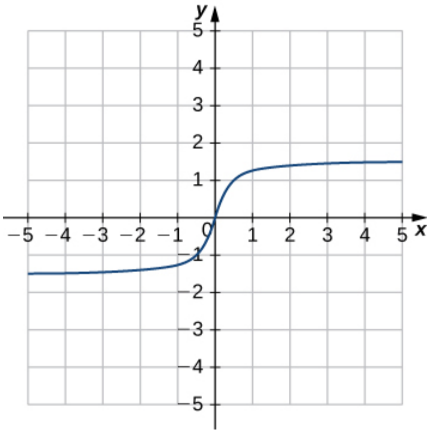

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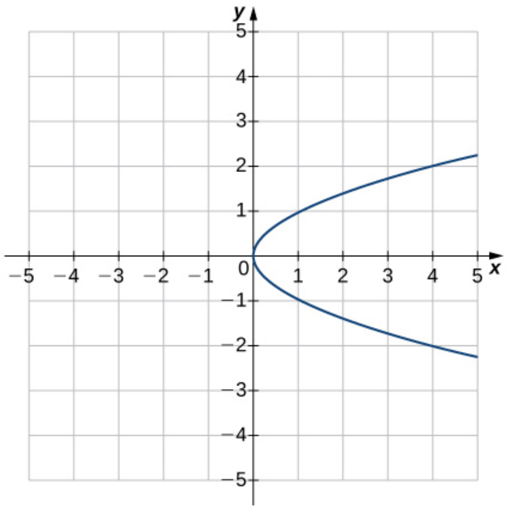
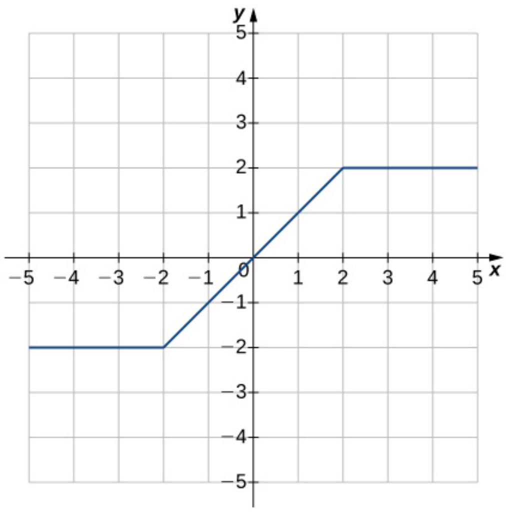


33.



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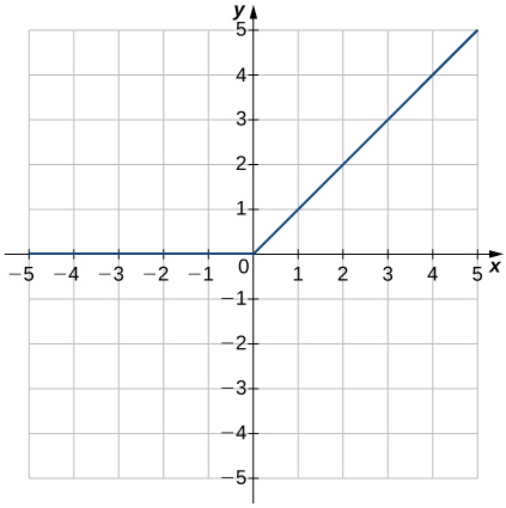

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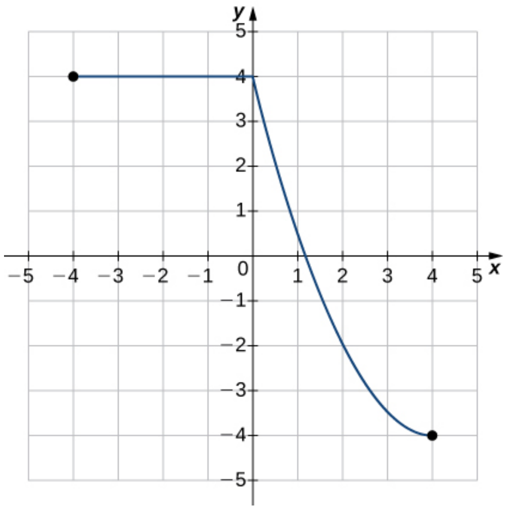


For the following exercises, for each pair of functions, find a. $f+g$ b. $f-g$ c. $f\cdot g$ d. f/g. Determine the domain of each of these new functions.

36. $f(x)=3x+4,\ g(x)=x-2$

37. $f(x)=x-8,\ g(x)=5x^2$

38. $f(x)=3x^2+4x+1,\ g(x)=x+1$

39. $f(x)=9-x^2,\ g(x)=x^2-2x-3$

40. $f(x)=\sqrt{x},\ g(x)=x-2$

41. $f(x)=6+\frac{1}{x},\ g(x)=\frac{1}{x}$

For the following exercises, for each pair of functions, find

a. $(f \circ g)(x)$ and b. $(g \circ f)(x)$ Simplify the results. Find the domain of each of the results.

42. $f(x) = 3x,\ g(x) = x + 5$

43. $f(x) = x + 4,\ g(x) = 4x - 1$

44. $f(x) = 2x + 4,\ g(x) = x^2 - 2$

45. $f(x) = x^2 + 7,\ g(x) = x^2 - 3$

46. $f(x) = \sqrt{x},\ g(x) = x + 9$

47. $f(x) = \frac{3}{2x + 1},\ g(x) = \frac{2}{x}$

48. $f(x) = |x + 1|,\ g(x) = x^2 + x - 4$

49. The table below lists the NBA championship winners for the years 2001 to 2012.

Year	Winner
2001	LA Lakers
2002	LA Lakers
2003	San Antonio Spurs
2004	Detroit Pistons
2005	San Antonio Spurs
2006	Miami Heat
2007	San Antonio Spurs
2008	Boston Celtics
2009	LA Lakers
2010	LA Lakers
2011	Dallas Mavericks
2012	Miami Heat

a. Consider the relation in which the domain values are the years 2001 to 2012 and the range is the corresponding winner. Is this relation a function? Explain why or why not.
b. Consider the relation where the domain values are the winners and the range is the corresponding years. Is this relation a function? Explain why or why not.

50. **[T]** The area A of a square depends on the length of the side s.
a. Write a function $A(s)$ for the area of a square.
b. Find and interpret $A(6.5)$.
c. Find the exact and the two-significant-digit approximation to the length of the sides of a square with area 56 square units.

51. **[T]** The volume of a cube depends on the length of the sides s.
   a. Write a function $V(s)$ for the volume of a cube.
   b. Find and interpret $V(11.8)$.

52. **[T]** A rental car company rents cars for a flat fee of \$20 and an hourly charge of \$10.25. Therefore, the total cost C to rent a car is a function of the hours t the car is rented plus the flat fee.
   a. Write the formula for the function that models this situation.
   b. Find the total cost to rent a car for 2 days and 7 hours.
   c. Determine how long the car was rented if the bill is \$432.73.

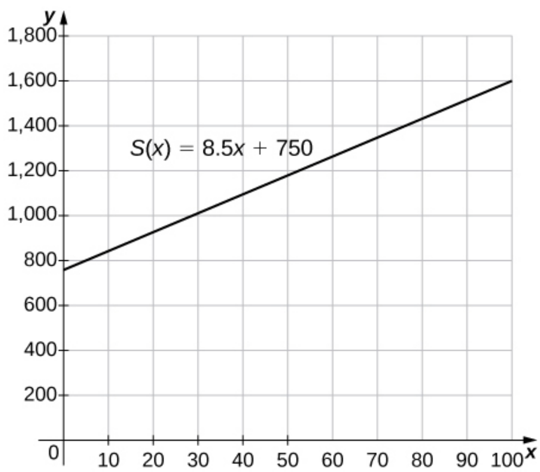
53. **[T]** A vehicle has a 20-gal tank and gets 15 mpg. The number of miles N that can be driven depends on the amount of gas x in the tank.
   a. Write a formula that models this situation.
   b. Determine the number of miles the vehicle can travel on (i) a full tank of gas and (ii) 3/4 of a tank of gas.
   c. Determine the domain and range of the function.
   d. Determine how many times the driver had to stop for gas if she has driven a total of 578 mi.

54. **[T]** The volume V of a sphere depends on the length of its radius as $V = (4/3)\pi r^3$. Because Earth is not a perfect sphere, we can use the *mean radius* when measuring from the center to its surface. The mean radius is the average distance from the physical center to the surface, based on a large number of samples. Find the volume of Earth with mean radius 6.371×10^6 m.

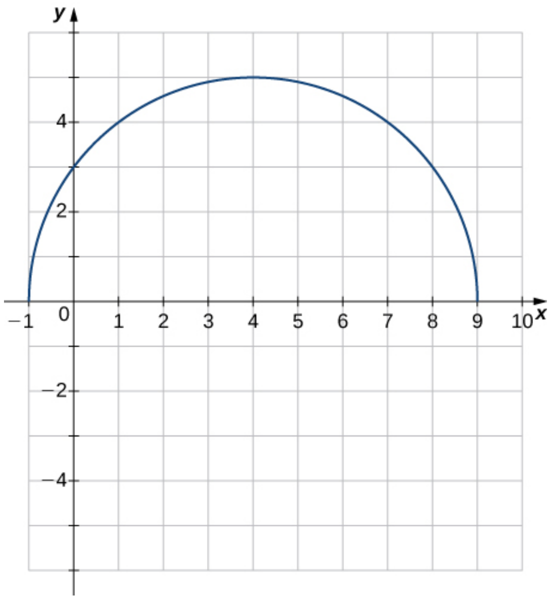
55. **[T]** A certain bacterium grows in culture in a circular region. The radius of the circle, measured in centimeters, is given by $r(t) = 6 - \left[5/\left(t^2 + 1\right)\right]$, where t is time measured in hours since a circle of a 1-cm radius of the bacterium was put into the culture.
   a. Express the area of the bacteria as a function of time.
   b. Find the exact and approximate area of the bacterial culture in 3 hours.
   c. Express the circumference of the bacteria as a function of time.
   d. Find the exact and approximate circumference of the bacteria in 3 hours.

56. **[T]** An American tourist visits Paris and must convert U.S. dollars to Euros, which can be done using the function $E(x) = 0.79x$, where x is the number of U.S. dollars and $E(x)$ is the equivalent number of Euros. Since conversion rates fluctuate, when the tourist returns to the United States 2 weeks later, the conversion from Euros to U.S. dollars is $D(x) = 1.245x$, where x is the number of Euros and $D(x)$ is the equivalent number of U.S. dollars.
   a. Find the composite function that converts directly from U.S. dollars to U.S. dollars via Euros. Did this tourist lose value in the conversion process?
   b. Use (a) to determine how many U.S. dollars the tourist would get back at the end of her trip if she converted an extra \$200 when she arrived in Paris.

57. **[T]** The manager at a skateboard shop pays his workers a monthly salary S of \$750 plus a commission of \$8.50 for each skateboard they sell.
   a. Write a function $y = S(x)$ that models a worker's monthly salary based on the number of skateboards x he or she sells.
   b. Find the approximate monthly salary when a worker sells 25, 40, or 55 skateboards.
   c. Use the INTERSECT feature on a graphing calculator to determine the number of skateboards that must be sold for a worker to earn a monthly income of \$1400. (*Hint*: Find the intersection of the function and the line $y = 1400$.)



58. **[T]** Use a graphing calculator to graph the half-circle $y = \sqrt{25 - (x - 4)^2}$. Then, use the INTERCEPT feature to find the value of both the x- and y-intercepts.



1.2 | Basic Classes of Functions

Learning Objectives

1.2.1 Calculate the slope of a linear function and interpret its meaning.
1.2.2 Recognize the degree of a polynomial.
1.2.3 Find the roots of a quadratic polynomial.
1.2.4 Describe the graphs of basic odd and even polynomial functions.
1.2.5 Identify a rational function.
1.2.6 Describe the graphs of power and root functions.
1.2.7 Explain the difference between algebraic and transcendental functions.
1.2.8 Graph a piecewise-defined function.
1.2.9 Sketch the graph of a function that has been shifted, stretched, or reflected from its initial graph position.

We have studied the general characteristics of functions, so now let's examine some specific classes of functions. We begin by reviewing the basic properties of linear and quadratic functions, and then generalize to include higher-degree polynomials. By combining root functions with polynomials, we can define general algebraic functions and distinguish them from the transcendental functions we examine later in this chapter. We finish the section with examples of piecewise-defined functions and take a look at how to sketch the graph of a function that has been shifted, stretched, or reflected from its initial form.

Linear Functions and Slope

The easiest type of function to consider is a **linear function**. Linear functions have the form $f(x) = ax + b$, where a and b are constants. In **Figure 1.15**, we see examples of linear functions when a is positive, negative, and zero. Note that if $a > 0$, the graph of the line rises as x increases. In other words, $f(x) = ax + b$ is increasing on $(-\infty, \infty)$. If $a < 0$, the graph of the line falls as x increases. In this case, $f(x) = ax + b$ is decreasing on $(-\infty, \infty)$. If $a = 0$, the line is horizontal.

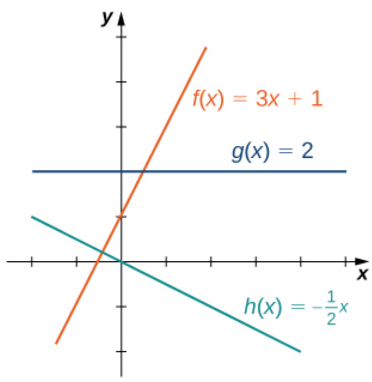


Figure 1.15 These linear functions are increasing or decreasing on (∞, ∞) and one function is a horizontal line.

As suggested by **Figure 1.15**, the graph of any linear function is a line. One of the distinguishing features of a line is its slope. The **slope** is the change in y for each unit change in x. The slope measures both the steepness and the direction of a line. If the slope is positive, the line points upward when moving from left to right. If the slope is negative, the line points downward when moving from left to right. If the slope is zero, the line is horizontal. To calculate the slope of a line, we need to determine the ratio of the change in y versus the change in x. To do so, we choose any two points (x_1, y_1) and (x_2, y_2) on the line and calculate $\frac{y_2 - y_1}{x_2 - x_1}$. In **Figure 1.16**, we see this ratio is independent of the points chosen.

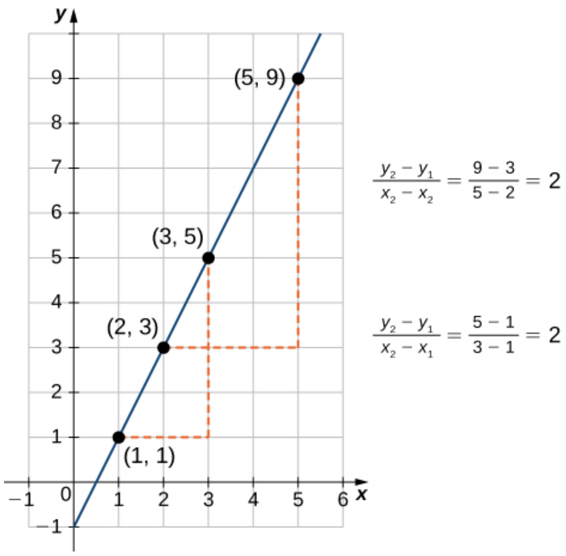


Figure 1.16 For any linear function, the slope $(y_2 - y_1)/(x_2 - x_1)$ is independent of the choice of points (x_1, y_1) and (x_2, y_2) on the line.

Definition

Consider line L passing through points (x_1, y_1) and (x_2, y_2). Let $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$ denote the changes in y and x, respectively. The **slope** of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}. \tag{1.3}$$

We now examine the relationship between slope and the formula for a linear function. Consider the linear function given by the formula $f(x) = ax + b$. As discussed earlier, we know the graph of a linear function is given by a line. We can use our definition of slope to calculate the slope of this line. As shown, we can determine the slope by calculating $(y_2 - y_1)/(x_2 - x_1)$ for any points (x_1, y_1) and (x_2, y_2) on the line. Evaluating the function f at $x = 0$, we see that $(0, b)$ is a point on this line. Evaluating this function at $x = 1$, we see that $(1, a + b)$ is also a point on this line. Therefore, the slope of this line is

$$\frac{(a + b) - b}{1 - 0} = a.$$

We have shown that the coefficient a is the slope of the line. We can conclude that the formula $f(x) = ax + b$ describes a line with slope a. Furthermore, because this line intersects the y-axis at the point $(0, b)$, we see that the y-intercept for this linear function is $(0, b)$. We conclude that the formula $f(x) = ax + b$ tells us the slope, a, and the y-intercept, $(0, b)$, for this line. Since we often use the symbol m to denote the slope of a line, we can write

$$f(x) = mx + b$$

to denote the **slope-intercept form** of a linear function.

Sometimes it is convenient to express a linear function in different ways. For example, suppose the graph of a linear function passes through the point (x_1, y_1) and the slope of the line is m. Since any other point $(x, f(x))$ on the graph of f must satisfy the equation

$$m = \frac{f(x) - y_1}{x - x_1},$$

this linear function can be expressed by writing

$$f(x) - y_1 = m(x - x_1).$$

We call this equation the **point-slope equation** for that linear function.

Since every nonvertical line is the graph of a linear function, the points on a nonvertical line can be described using the slope-intercept or point-slope equations. However, a vertical line does not represent the graph of a function and cannot be expressed in either of these forms. Instead, a vertical line is described by the equation $x = k$ for some constant k. Since neither the slope-intercept form nor the point-slope form allows for vertical lines, we use the notation

$$ax + by = c,$$

where a, b are both not zero, to denote the **standard form of a line**.

Definition

Consider a line passing through the point (x_1, y_1) with slope m. The equation

$$y - y_1 = m(x - x_1) \tag{1.4}$$

is the **point-slope equation** for that line.

Consider a line with slope m and y-intercept $(0, b)$. The equation

$$y = mx + b \tag{1.5}$$

is an equation for that line in **slope-intercept form**.

The **standard form of a line** is given by the equation

$$ax + by = c, \tag{1.6}$$

where a and b are both not zero. This form is more general because it allows for a vertical line, $x = k$.

Example 1.12

Finding the Slope and Equations of Lines

Consider the line passing through the points $(11, -4)$ and $(-4, 5)$, as shown in **Figure 1.17**.

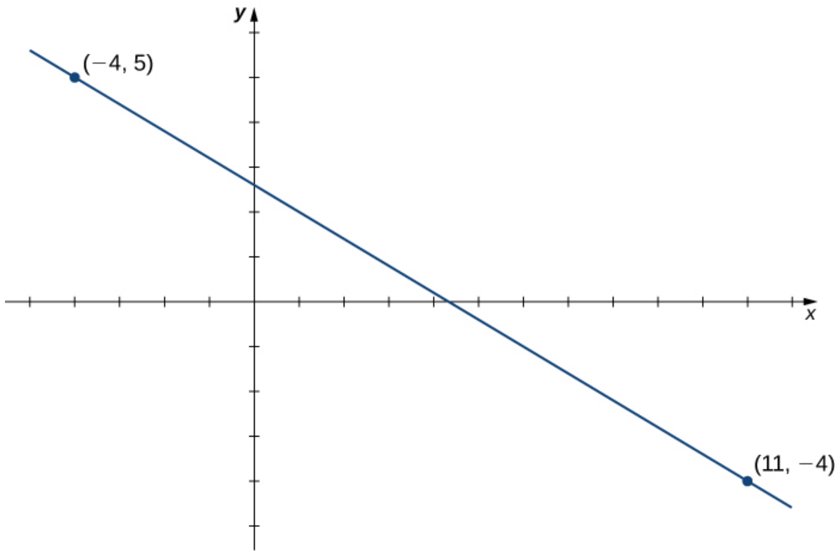


Figure 1.17 Finding the equation of a linear function with a graph that is a line between two given points.

a. Find the slope of the line.
b. Find an equation for this linear function in point-slope form.
c. Find an equation for this linear function in slope-intercept form.

Solution

a. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-4)}{-4 - 11} = -\frac{9}{15} = -\frac{3}{5}.$$

b. To find an equation for the linear function in point-slope form, use the slope $m = -3/5$ and choose any point on the line. If we choose the point $(11, -4)$, we get the equation

$$f(x) + 4 = -\frac{3}{5}(x - 11).$$

c. To find an equation for the linear function in slope-intercept form, solve the equation in part b. for $f(x)$. When we do this, we get the equation

$$f(x) = -\frac{3}{5}x + \frac{13}{5}.$$

1.9 Consider the line passing through points $(-3, 2)$ and $(1, 4)$. Find the slope of the line.

Find an equation of that line in point-slope form. Find an equation of that line in slope-intercept form.

Example 1.13

A Linear Distance Function

Jessica leaves her house at 5:50 a.m. and goes for a 9-mile run. She returns to her house at 7:08 a.m. Answer the following questions, assuming Jessica runs at a constant pace.

a. Describe the distance D (in miles) Jessica runs as a linear function of her run time t (in minutes).

b. Sketch a graph of D.

c. Interpret the meaning of the slope.

Solution

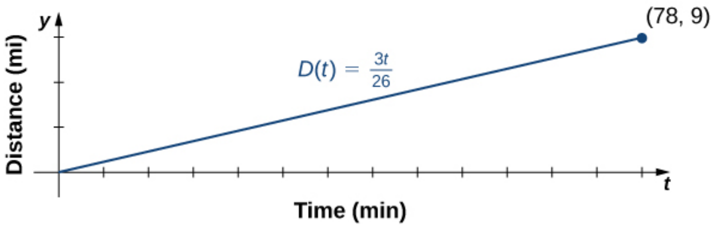
a. At time $t = 0,$ Jessica is at her house, so $D(0) = 0.$ At time $t = 78$ minutes, Jessica has finished running 9 mi, so $D(78) = 9.$ The slope of the linear function is

$$m = \frac{9 - 0}{78 - 0} = \frac{3}{26}.$$

The y-intercept is $(0, 0),$ so the equation for this linear function is

$$D(t) = \frac{3}{26}t.$$

b. To graph $D,$ use the fact that the graph passes through the origin and has slope $m = 3/26.$



c. The slope $m = 3/26 \approx 0.115$ describes the distance (in miles) Jessica runs per minute, or her average velocity.

Polynomials

A linear function is a special type of a more general class of functions: polynomials. A **polynomial function** is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{1.7}$$

for some integer $n \geq 0$ and constants $a_n, a_{n-1}, \ldots, a_0,$ where $a_n \neq 0.$ In the case when $n = 0,$ we allow for $a_0 = 0;$ if $a_0 = 0,$ the function $f(x) = 0$ is called the *zero function*. The value n is called the **degree** of the polynomial; the constant a_n is called the *leading coefficient*. A linear function of the form $f(x) = mx + b$ is a polynomial of degree 1 if $m \neq 0$ and degree 0 if $m = 0.$ A polynomial of degree 0 is also called a *constant function*. A polynomial function of degree 2 is called a **quadratic function**. In particular, a quadratic function has the form $f(x) = ax^2 + bx + c,$ where $a \neq 0.$ A polynomial function of degree 3 is called a **cubic function**.

Power Functions

Some polynomial functions are power functions. A **power function** is any function of the form $f(x) = ax^b,$ where a and b are any real numbers. The exponent in a power function can be any real number, but here we consider the case when the exponent is a positive integer. (We consider other cases later.) If the exponent is a positive integer, then $f(x) = ax^n$ is a polynomial. If n is even, then $f(x) = ax^n$ is an even function because $f(-x) = a(-x)^n = ax^n$ if n is even. If n is odd, then $f(x) = ax^n$ is an odd function because $f(-x) = a(-x)^n = -ax^n$ if n is odd (Figure 1.18).

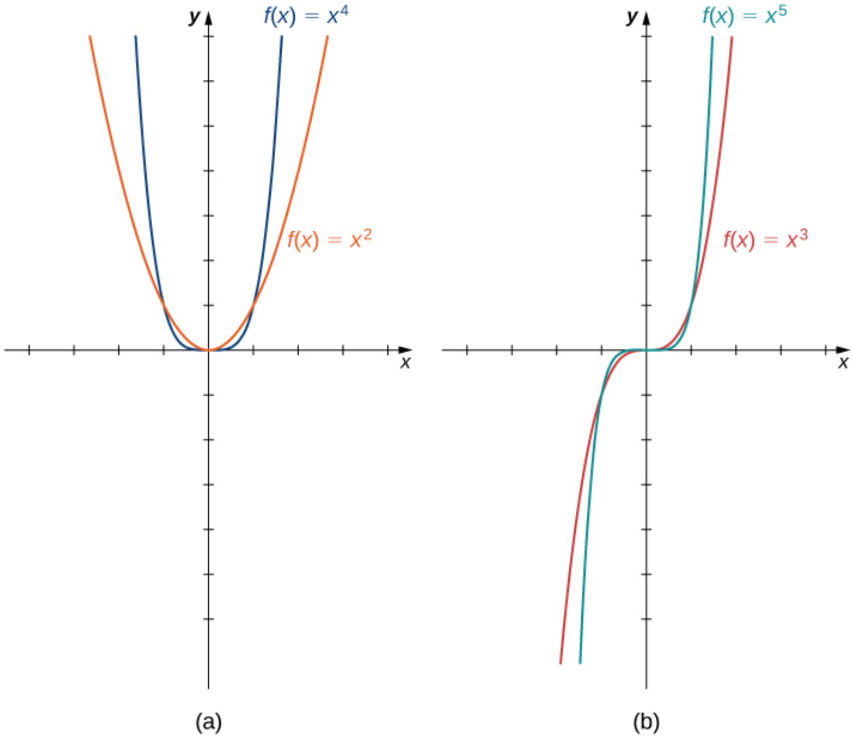


Figure 1.18 (a) For any even integer n, $f(x) = ax^n$ is an even function. (b) For any odd integer n, $f(x) = ax^n$ is an odd function.

Behavior at Infinity

To determine the behavior of a function f as the inputs approach infinity, we look at the values $f(x)$ as the inputs, x, become larger. For some functions, the values of $f(x)$ approach a finite number. For example, for the function $f(x) = 2 + 1/x$, the values $1/x$ become closer and closer to zero for all values of x as they get larger and larger. For this function, we say "$f(x)$ approaches two as x goes to infinity," and we write $f(x) \to 2$ as $x \to \infty$. The line $y = 2$ is a horizontal asymptote for the function $f(x) = 2 + 1/x$ because the graph of the function gets closer to the line as x gets larger.

For other functions, the values $f(x)$ may not approach a finite number but instead may become larger for all values of x as they get larger. In that case, we say "$f(x)$ approaches infinity as x approaches infinity," and we write $f(x) \to \infty$ as $x \to \infty$. For example, for the function $f(x) = 3x^2$, the outputs $f(x)$ become larger as the inputs x get larger. We can conclude that the function $f(x) = 3x^2$ approaches infinity as x approaches infinity, and we write $3x^2 \to \infty$ as $x \to \infty$. The behavior as $x \to -\infty$ and the meaning of $f(x) \to -\infty$ as $x \to \infty$ or $x \to -\infty$ can be defined similarly. We can describe what happens to the values of $f(x)$ as $x \to \infty$ and as $x \to -\infty$ as the *end behavior* of the function.

To understand the end behavior for polynomial functions, we can focus on quadratic and cubic functions. The behavior for higher-degree polynomials can be analyzed similarly. Consider a quadratic function $f(x) = ax^2 + bx + c$. If $a > 0$, the values $f(x) \to \infty$ as $x \to \pm\infty$. If $a < 0$, the values $f(x) \to -\infty$ as $x \to \pm\infty$. Since the graph of a quadratic function is a parabola, the parabola opens upward if $a > 0$; the parabola opens downward if $a < 0$. (See **Figure 1.19**(a).)

Now consider a cubic function $f(x) = ax^3 + bx^2 + cx + d$. If $a > 0$, then $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. If $a < 0$, then $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$. As we can see from both of these graphs, the leading term of the polynomial determines the end behavior. (See **Figure 1.19**(b).)

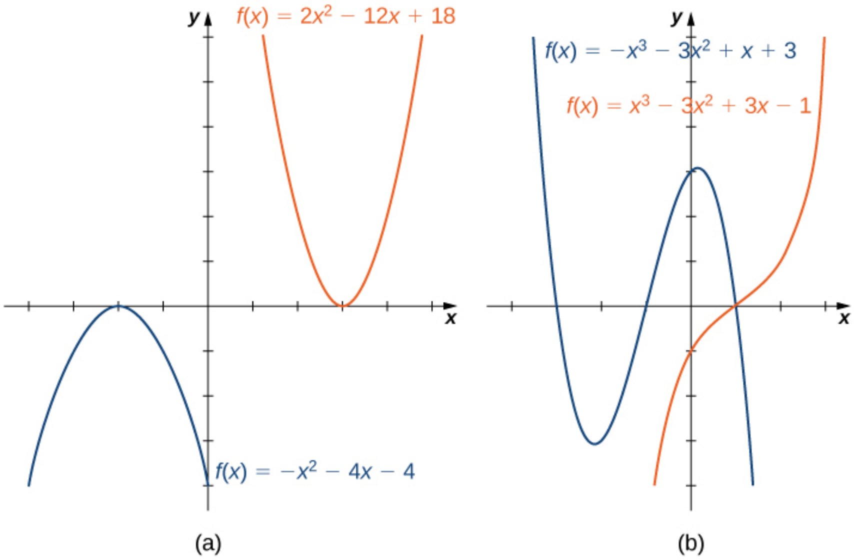


Figure 1.19 (a) For a quadratic function, if the leading coefficient $a > 0,$ the parabola opens upward. If $a < 0,$ the parabola opens downward. (b) For a cubic function $f,$ if the leading coefficient $a > 0,$ the values $f(x) \to \infty$ as $x \to \infty$ and the values $f(x) \to -\infty$ as $x \to -\infty.$ If the leading coefficient $a < 0,$ the opposite is true.

Zeros of Polynomial Functions

Another characteristic of the graph of a polynomial function is where it intersects the x-axis. To determine where a function f intersects the x-axis, we need to solve the equation $f(x) = 0$ for x. In the case of the linear function $f(x) = mx + b,$ the x-intercept is given by solving the equation $mx + b = 0.$ In this case, we see that the x-intercept is given by $(-b/m,\ 0).$ In the case of a quadratic function, finding the x-intercept(s) requires finding the zeros of a quadratic equation: $ax^2 + bx + c = 0.$ In some cases, it is easy to factor the polynomial $ax^2 + bx + c$ to find the zeros. If not, we make use of the quadratic formula.

Rule: The Quadratic Formula

Consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a \neq 0.$ The solutions of this equation are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{1.8}$$

If the discriminant $b^2 - 4ac > 0,$ this formula tells us there are two real numbers that satisfy the quadratic equation. If $b^2 - 4ac = 0,$ this formula tells us there is only one solution, and it is a real number. If $b^2 - 4ac < 0,$ no real numbers satisfy the quadratic equation.

In the case of higher-degree polynomials, it may be more complicated to determine where the graph intersects the x-axis. In some instances, it is possible to find the x-intercepts by factoring the polynomial to find its zeros. In other cases, it is impossible to calculate the exact values of the x-intercepts. However, as we see later in the text, in cases such as this, we can use analytical tools to approximate (to a very high degree) where the x-intercepts are located. Here we focus on the graphs of polynomials for which we can calculate their zeros explicitly.

Example 1.14

Graphing Polynomial Functions

For the following functions a. and b., i. describe the behavior of $f(x)$ as $x \to \pm\infty$, ii. find all zeros of f, and iii. sketch a graph of f.

a. $f(x) = -2x^2 + 4x - 1$

b. $f(x) = x^3 - 3x^2 - 4x$

Solution

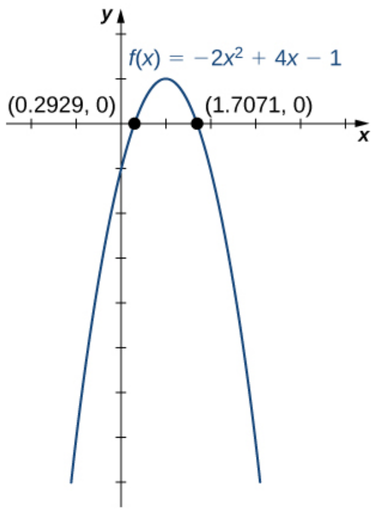
a. The function $f(x) = -2x^2 + 4x - 1$ is a quadratic function.

i. Because $a = -2 < 0$, as $x \to \pm\infty$, $f(x) \to -\infty$.

ii. To find the zeros of f, use the quadratic formula. The zeros are

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(-1)}}{2(-2)} = \frac{-4 \pm \sqrt{8}}{-4} = \frac{-4 \pm 2\sqrt{2}}{-4} = \frac{2 \pm \sqrt{2}}{2}.$$

iii. To sketch the graph of f, use the information from your previous answers and combine it with the fact that the graph is a parabola opening downward.



b. The function $f(x) = x^3 - 3x^2 - 4x$ is a cubic function.

i. Because $a = 1 > 0$, as $x \to \infty$, $f(x) \to \infty$. As $x \to -\infty$, $f(x) \to -\infty$.

ii. To find the zeros of f, we need to factor the polynomial. First, when we factor x out of all the terms, we find

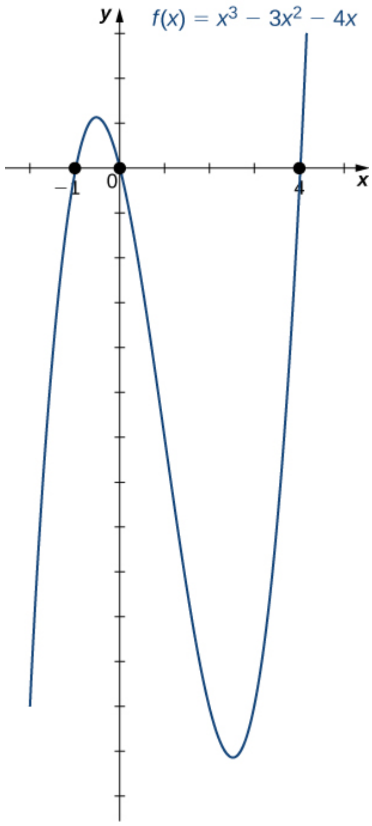
$$f(x) = x(x^2 - 3x - 4).$$

Then, when we factor the quadratic function $x^2 - 3x - 4$, we find

$$f(x) = x(x - 4)(x + 1).$$

Therefore, the zeros of f are $x = 0,\ 4,\ -1.$

iii. Combining the results from parts i. and ii., draw a rough sketch of f.



1.10 Consider the quadratic function $f(x) = 3x^2 - 6x + 2.$ Find the zeros of f. Does the parabola open upward or downward?

Mathematical Models

A large variety of real-world situations can be described using **mathematical models**. A mathematical model is a method of simulating real-life situations with mathematical equations. Physicists, engineers, economists, and other researchers develop models by combining observation with quantitative data to develop equations, functions, graphs, and other mathematical tools to describe the behavior of various systems accurately. Models are useful because they help predict future outcomes. Examples of mathematical models include the study of population dynamics, investigations of weather patterns, and predictions of product sales.

As an example, let's consider a mathematical model that a company could use to describe its revenue for the sale of a particular item. The amount of revenue R a company receives for the sale of n items sold at a price of p dollars per item is described by the equation $R = p \cdot n.$ The company is interested in how the sales change as the price of the item changes. Suppose the data in Table 1.6 show the number of units a company sells as a function of the price per item.

p	6	8	10	12	14
n	19.4	18.5	16.2	13.8	12.2

Table 1.6 Number of Units Sold n (in Thousands) as a Function of Price per Unit p (in Dollars)

In **Figure 1.20**, we see the graph the number of units sold (in thousands) as a function of price (in dollars). We note from the shape of the graph that the number of units sold is likely a linear function of price per item, and the data can be closely approximated by the linear function $n = -1.04p + 26$ for $0 \le p \le 25,$ where n predicts the number of units sold in thousands. Using this linear function, the revenue (in thousands of dollars) can be estimated by the quadratic function

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p$$

for $0 \le p \le 25.$ In **Example 1.15**, we use this quadratic function to predict the amount of revenue the company receives depending on the price the company charges per item. Note that we cannot conclude definitively the actual number of units sold for values of $p,$ for which no data are collected. However, given the other data values and the graph shown, it seems reasonable that the number of units sold (in thousands) if the price charged is p dollars may be close to the values predicted by the linear function $n = -1.04p + 26.$

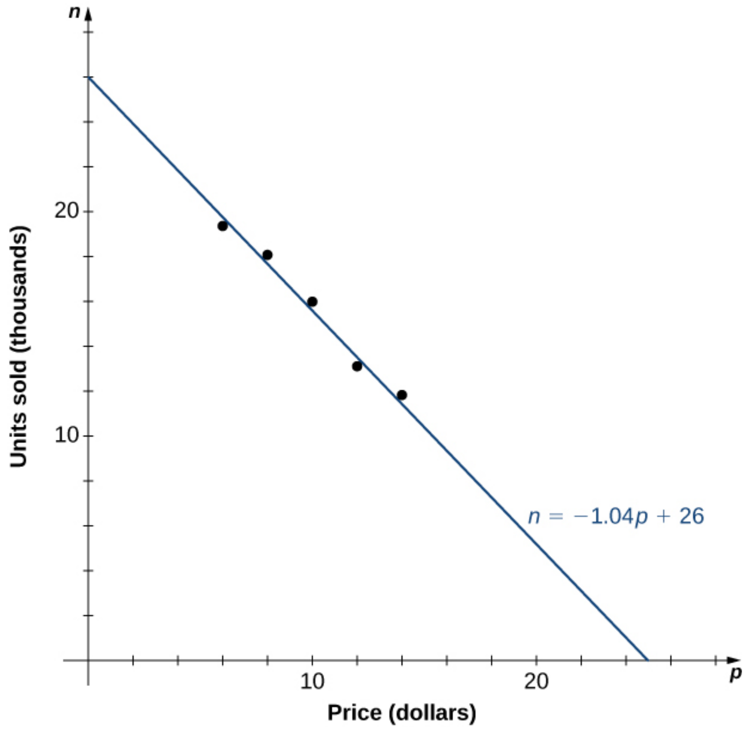


Figure 1.20 The data collected for the number of items sold as a function of price is roughly linear. We use the linear function $n = -1.04p + 26$ to estimate this function.

Example 1.15

Maximizing Revenue

A company is interested in predicting the amount of revenue it will receive depending on the price it charges for a particular item. Using the data from **Table 1.6**, the company arrives at the following quadratic function to model revenue R (in thousands of dollars) as a function of price per item p:

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p$$

for $0 \le p \le 25$.

a. Predict the revenue if the company sells the item at a price of $p = \$5$ and $p = \$17$.

b. Find the zeros of this function and interpret the meaning of the zeros.

c. Sketch a graph of R.

d. Use the graph to determine the value of p that maximizes revenue. Find the maximum revenue.

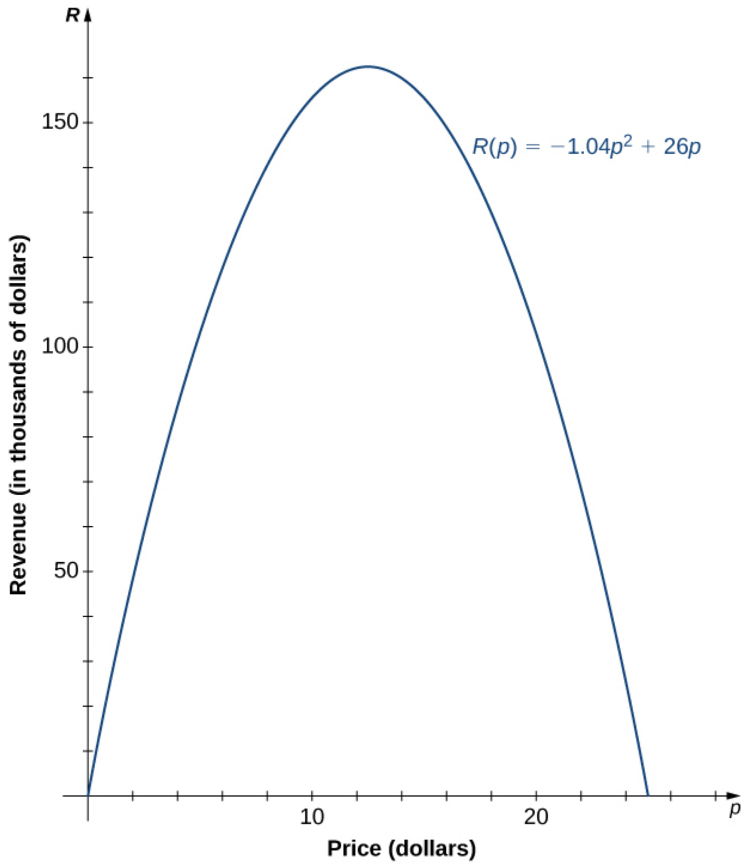
Solution

a. Evaluating the revenue function at $p = 5$ and $p = 17$, we can conclude that

$$R(5) = -1.04(5)^2 + 26(5) = 104, \text{ so revenue } = \$104{,}000;$$
$$R(17) = -1.04(17)^2 + 26(17) = 141.44, \text{ so revenue } = \$141{,}440.$$

b. The zeros of this function can be found by solving the equation $-1.04p^2 + 26p = 0$. When we factor the quadratic expression, we get $p(-1.04p + 26) = 0$. The solutions to this equation are given by $p = 0, 25$. For these values of p, the revenue is zero. When $p = \$0$, the revenue is zero because the company is giving away its merchandise for free. When $p = \$25$, the revenue is zero because the price is too high, and no one will buy any items.

c. Knowing the fact that the function is quadratic, we also know the graph is a parabola. Since the leading coefficient is negative, the parabola opens downward. One property of parabolas is that they are symmetric about the axis, so since the zeros are at $p = 0$ and $p = 25$, the parabola must be symmetric about the line halfway between them, or $p = 12.5$.



d. The function is a parabola with zeros at $p = 0$ and $p = 25,$ and it is symmetric about the line $p = 12.5,$ so the maximum revenue occurs at a price of $p = \$12.50$ per item. At that price, the revenue is $R(p) = -1.04(12.5)^2 + 26(12.5) = \$162,\ 500.$

Algebraic Functions

By allowing for quotients and fractional powers in polynomial functions, we create a larger class of functions. An **algebraic function** is one that involves addition, subtraction, multiplication, division, rational powers, and roots. Two types of algebraic functions are rational functions and root functions.

Just as rational numbers are quotients of integers, rational functions are quotients of polynomials. In particular, a **rational function** is any function of the form $f(x) = p(x)/q(x),$ where $p(x)$ and $q(x)$ are polynomials. For example,

$$f(x) = \frac{3x - 1}{5x + 2} \qquad \text{and} \qquad g(x) = \frac{4}{x^2 + 1}$$

are rational functions. A **root function** is a power function of the form $f(x) = x^{1/n},$ where n is a positive integer greater than one. For example, $f(x) = x^{1/2} = \sqrt{x}$ is the square-root function and $g(x) = x^{1/3} = \sqrt[3]{x}$ is the cube-root function. By allowing for compositions of root functions and rational functions, we can create other algebraic functions. For example, $f(x) = \sqrt{4 - x^2}$ is an algebraic function.

Example 1.16

Finding Domain and Range for Algebraic Functions

For each of the following functions, find the domain and range.

a. $f(x) = \dfrac{3x-1}{5x+2}$

b. To find the domain of f, we need $4 - x^2 \geq 0$. Or, $4 \geq x^2$ Or $x^2 \leq 4$, the solution to which is $-2 \leq x \leq 2$. Therefore, the domain is $\{x| -2 \leq x \leq 2\}$. If $-2 \leq x \leq 2$, then $0 \leq 4 - x^2 \leq 4$. Therefore, $0 \leq \sqrt{4 - x^2} \leq 2$ and the range of f is $\{y|0 \leq x \leq 2\}$.

Solution

a. It is not possible to divide by zero, so the domain is the set of real numbers x such that $x \neq -2/5$. To find the range, we need to find the values y for which there exists a real number x such that

$$y = \frac{3x-1}{5x+2}.$$

When we multiply both sides of this equation by $5x+2$, we see that x must satisfy the equation

$$5xy + 2y = 3x - 1.$$

From this equation, we can see that x must satisfy

$$2y + 1 = x(3 - 5y).$$

If $y = 3/5$, this equation has no solution. On the other hand, as long as $y \neq 3/5$,

$$x = \frac{2y+1}{3-5y}$$

satisfies this equation. We can conclude that the range of f is $\{y|y \neq 3/5\}$.

b. To find the domain of f, we need $4 - x^2 \geq 0$. When we factor, we write $4 - x^2 = (2-x)(2+x) \geq 0$. This inequality holds if and only if both terms are positive or both terms are negative. For both terms to be positive, we need to find x such that

$$2 - x \geq 0 \quad \text{and} \quad 2 + x \geq 0.$$

These two inequalities reduce to $2 \geq x$ and $x \geq -2$. Therefore, the set $\{x| -2 \leq x \leq 2\}$ must be part of the domain. For both terms to be negative, we need

$$2 - x \leq 0 \quad \text{and} \quad 2 + x \geq 0.$$

These two inequalities also reduce to $2 \leq x$ and $x \geq -2$. There are no values of x that satisfy both of these inequalities. Thus, we can conclude the domain of this function is $\{x| -2 \leq x \leq 2\}$.

If $-2 \leq x \leq 2$, then $0 \leq 4 - x^2 \leq 4$. Therefore, $0 \leq \sqrt{4 - x^2} \leq 2$, and the range of f is $\{y|0 \leq y \leq 2\}$.

1.11 Find the domain and range for the function $f(x) = (5x + 2)/(2x - 1)$.

The root functions $f(x) = x^{1/n}$ have defining characteristics depending on whether n is odd or even. For all even integers $n \geq 2$, the domain of $f(x) = x^{1/n}$ is the interval $[0, \infty)$. For all odd integers $n \geq 1$, the domain of $f(x) = x^{1/n}$ is the set of all real numbers. Since $x^{1/n} = (-x)^{1/n}$ for odd integers n, $f(x) = x^{1/n}$ is an odd function if n is odd. See the graphs of root functions for different values of n in **Figure 1.21**.

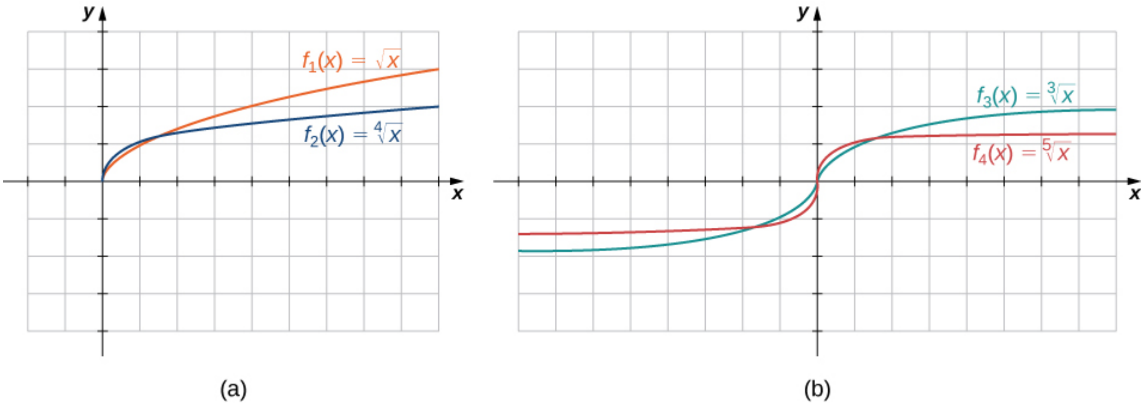


Figure 1.21 (a) If n is even, the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$. (b) If n is odd, the domain of $f(x) = \sqrt[n]{x}$ is $(-\infty, \infty)$ and the function $f(x) = \sqrt[n]{x}$ is an odd function.

Example 1.17

Finding Domains for Algebraic Functions

For each of the following functions, determine the domain of the function.

a. $f(x) = \dfrac{3}{x^2 - 1}$

b. $f(x) = \dfrac{2x + 5}{3x^2 + 4}$

c. $f(x) = \sqrt{4 - 3x}$

d. $f(x) = \sqrt[3]{2x - 1}$

Solution

a. You cannot divide by zero, so the domain is the set of values x such that $x^2 - 1 \neq 0$. Therefore, the domain is $\{x | x \neq \pm 1\}$.

b. You need to determine the values of x for which the denominator is zero. Since $3x^2 + 4 \geq 4$ for all real numbers x, the denominator is never zero. Therefore, the domain is $(-\infty, \infty)$.

c. Since the square root of a negative number is not a real number, the domain is the set of values x for which $4 - 3x \geq 0$. Therefore, the domain is $\{x | x \leq 4/3\}$.

d. The cube root is defined for all real numbers, so the domain is the interval $(-\infty, \infty)$.

1.12 Find the domain for each of the following functions: $f(x) = (5 - 2x)/(x^2 + 2)$ and $g(x) = \sqrt{5x - 1}$.

Transcendental Functions

Thus far, we have discussed algebraic functions. Some functions, however, cannot be described by basic algebraic operations. These functions are known as **transcendental functions** because they are said to "transcend," or go beyond, algebra. The most common transcendental functions are trigonometric, exponential, and logarithmic functions. A *trigonometric function* relates the ratios of two sides of a right triangle. They are $\sin x, \cos x, \tan x, \cot x, \sec x$, and $\csc x$. (We discuss trigonometric functions later in the chapter.) An exponential function is a function of the form $f(x) = b^x$, where the base $b > 0, b \neq 1$. A **logarithmic function** is a function of the form $f(x) = \log_b(x)$ for some constant $b > 0, b \neq 1$, where $\log_b(x) = y$ if and only if $b^y = x$. (We also discuss exponential and logarithmic functions later in the chapter.)

Example 1.18

Classifying Algebraic and Transcendental Functions

Classify each of the following functions, a. through c., as algebraic or transcendental.

a. $f(x) = \dfrac{\sqrt{x^3 + 1}}{4x + 2}$

b. $f(x) = 2^{x^2}$

c. $f(x) = \sin(2x)$

Solution

a. Since this function involves basic algebraic operations only, it is an algebraic function.

b. This function cannot be written as a formula that involves only basic algebraic operations, so it is transcendental. (Note that algebraic functions can only have powers that are rational numbers.)

c. As in part b., this function cannot be written using a formula involving basic algebraic operations only; therefore, this function is transcendental.

1.13 Is $f(x) = x/2$ an algebraic or a transcendental function?

Piecewise-Defined Functions

Sometimes a function is defined by different formulas on different parts of its domain. A function with this property is known as a **piecewise-defined function**. The absolute value function is an example of a piecewise-defined function because the formula changes with the sign of x:

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}.$$

Other piecewise-defined functions may be represented by completely different formulas, depending on the part of the domain in which a point falls. To graph a piecewise-defined function, we graph each part of the function in its respective domain, on the same coordinate system. If the formula for a function is different for $x < a$ and $x > a$, we need to pay special attention to what happens at $x = a$ when we graph the function. Sometimes the graph needs to include an open or closed circle to indicate the value of the function at $x = a$. We examine this in the next example.

Example 1.19

Graphing a Piecewise-Defined Function

Sketch a graph of the following piecewise-defined function:

$$f(x) = \begin{cases} x + 3, & x < 1 \\ (x - 2)^2, & x \geq 1 \end{cases}.$$

Solution

Graph the linear function $y = x + 3$ on the interval $(-\infty, 1)$ and graph the quadratic function $y = (x - 2)^2$ on the interval $[1, \infty)$. Since the value of the function at $x = 1$ is given by the formula $f(x) = (x - 2)^2$, we see that $f(1) = 1$. To indicate this on the graph, we draw a closed circle at the point $(1, 1)$. The value of the function is given by $f(x) = x + 2$ for all $x < 1$, but not at $x = 1$. To indicate this on the graph, we draw an open circle at $(1, 4)$.

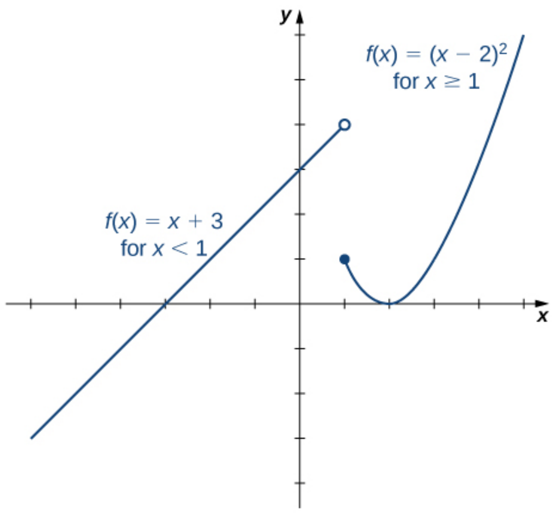


Figure 1.22 This piecewise-defined function is linear for $x < 1$ and quadratic for $x \geq 1$.

1.14 Sketch a graph of the function

$$f(x) = \begin{cases} 2 - x, & x \leq 2 \\ x + 2, & x > 2 \end{cases}.$$

Example 1.20

Parking Fees Described by a Piecewise-Defined Function

In a big city, drivers are charged variable rates for parking in a parking garage. They are charged $10 for the first hour or any part of the first hour and an additional $2 for each hour or part thereof up to a maximum of $30 for the day. The parking garage is open from 6 a.m. to 12 midnight.

a. Write a piecewise-defined function that describes the cost C to park in the parking garage as a function of hours parked x.

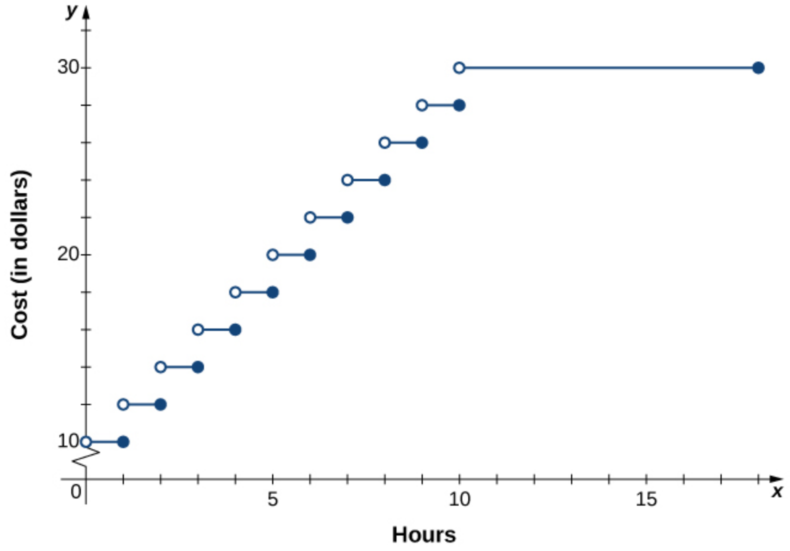
b. Sketch a graph of this function $C(x)$.

Solution

a. Since the parking garage is open 18 hours each day, the domain for this function is $\{x|0 < x \leq 18\}$. The cost to park a car at this parking garage can be described piecewise by the function

$$C(x) = \begin{cases} 10, & 0 < x \leq 1 \\ 12, & 1 < x \leq 2 \\ 14, & 2 < x \leq 3 \\ 16, & 3 < x \leq 4 \\ & \vdots \\ 30, & 10 < x \leq 18 \end{cases}.$$

b. The graph of the function consists of several horizontal line segments.



1.15 The cost of mailing a letter is a function of the weight of the letter. Suppose the cost of mailing a letter is $49¢$ for the first ounce and $21¢$ for each additional ounce. Write a piecewise-defined function describing the cost C as a function of the weight x for $0 < x \leq 3$, where C is measured in cents and x is measured in ounces.

Transformations of Functions

We have seen several cases in which we have added, subtracted, or multiplied constants to form variations of simple functions. In the previous example, for instance, we subtracted 2 from the argument of the function $y = x^2$ to get the function $f(x) = (x - 2)^2$. This subtraction represents a shift of the function $y = x^2$ two units to the right. A shift, horizontally or vertically, is a type of **transformation of a function**. Other transformations include horizontal and vertical scalings, and reflections about the axes.

A vertical shift of a function occurs if we add or subtract the same constant to each output y. For $c > 0$, the graph of $f(x) + c$ is a shift of the graph of $f(x)$ up c units, whereas the graph of $f(x) - c$ is a shift of the graph of $f(x)$ down c units. For example, the graph of the function $f(x) = x^3 + 4$ is the graph of $y = x^3$ shifted up 4 units; the graph of the function $f(x) = x^3 - 4$ is the graph of $y = x^3$ shifted down 4 units (**Figure 1.23**).

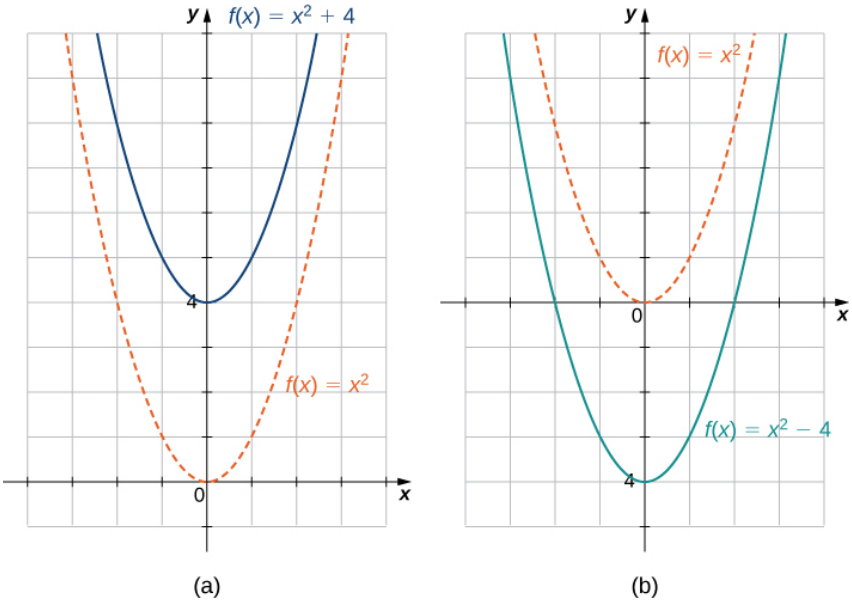


Figure 1.23 (a) For $c > 0$, the graph of $y = f(x) + c$ is a vertical shift up c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x) - c$ is a vertical shift down c units of the graph of $y = f(x)$.

A horizontal shift of a function occurs if we add or subtract the same constant to each input x. For $c > 0$, the graph of $f(x + c)$ is a shift of the graph of $f(x)$ to the left c units; the graph of $f(x - c)$ is a shift of the graph of $f(x)$ to the right c units. Why does the graph shift left when adding a constant and shift right when subtracting a constant? To answer this question, let's look at an example.

Consider the function $f(x) = |x + 3|$ and evaluate this function at $x - 3$. Since $f(x - 3) = |x|$ and $x - 3 < x$, the graph of $f(x) = |x + 3|$ is the graph of $y = |x|$ shifted left 3 units. Similarly, the graph of $f(x) = |x - 3|$ is the graph of $y = |x|$ shifted right 3 units (**Figure 1.24**).

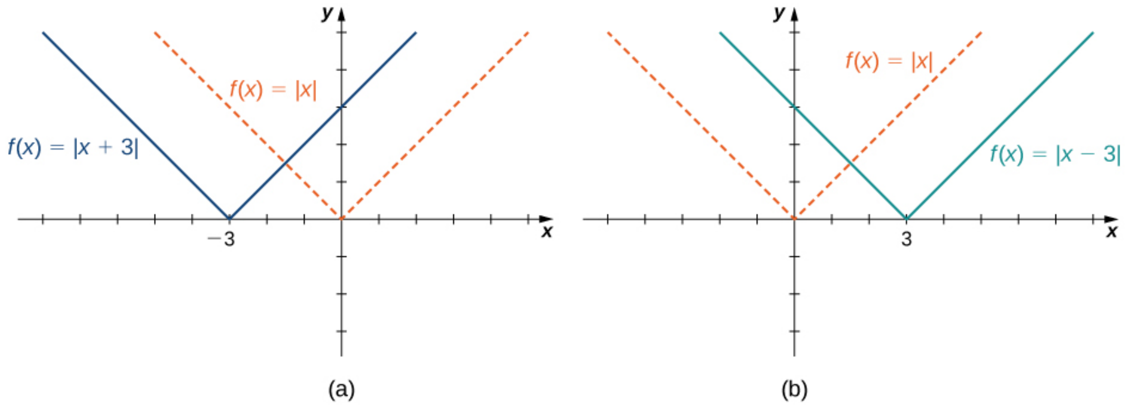


Figure 1.24 (a) For $c > 0,$ the graph of $y = f(x + c)$ is a horizontal shift left c units of the graph of $y = f(x)$. (b) For $c > 0,$ the graph of $y = f(x - c)$ is a horizontal shift right c units of the graph of $y = f(x)$.

A vertical scaling of a graph occurs if we multiply all outputs y of a function by the same positive constant. For $c > 0,$ the graph of the function $cf(x)$ is the graph of $f(x)$ scaled vertically by a factor of c. If $c > 1,$ the values of the outputs for the function $cf(x)$ are larger than the values of the outputs for the function $f(x)$; therefore, the graph has been stretched vertically. If $0 < c < 1,$ then the outputs of the function $cf(x)$ are smaller, so the graph has been compressed. For example, the graph of the function $f(x) = 3x^2$ is the graph of $y = x^2$ stretched vertically by a factor of 3, whereas the graph of $f(x) = x^2/3$ is the graph of $y = x^2$ compressed vertically by a factor of 3 (**Figure 1.25**).

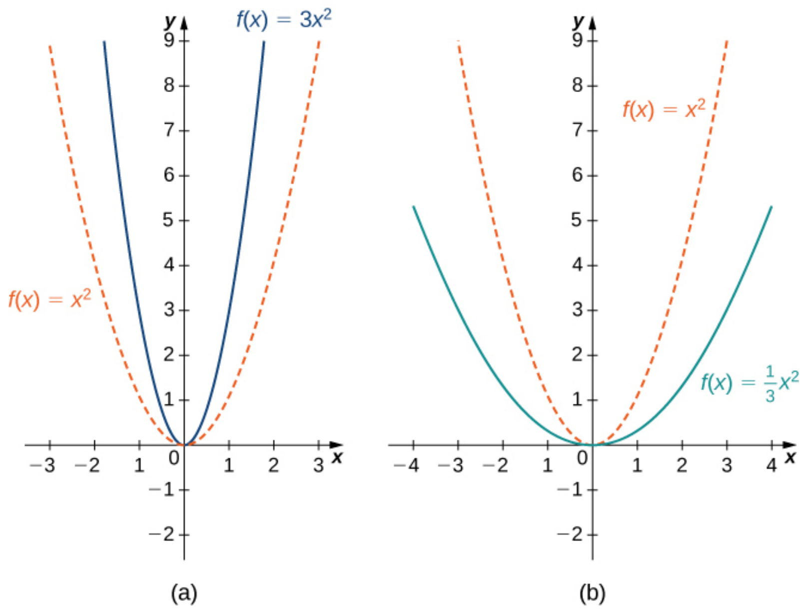


Figure 1.25 (a) If $c > 1,$ the graph of $y = cf(x)$ is a vertical stretch of the graph of $y = f(x)$. (b) If $0 < c < 1,$ the graph of $y = cf(x)$ is a vertical compression of the graph of $y = f(x)$.

The horizontal scaling of a function occurs if we multiply the inputs x by the same positive constant. For $c > 0,$ the graph of the function $f(cx)$ is the graph of $f(x)$ scaled horizontally by a factor of c. If $c > 1,$ the graph of $f(cx)$ is the graph of $f(x)$ compressed horizontally. If $0 < c < 1,$ the graph of $f(cx)$ is the graph of $f(x)$ stretched horizontally. For example, consider the function $f(x) = \sqrt{2x}$ and evaluate f at $x/2$. Since $f(x/2) = \sqrt{x}$, the graph of $f(x) = \sqrt{2x}$ is the graph of $y = \sqrt{x}$ compressed horizontally. The graph of $y = \sqrt{x/2}$ is a horizontal stretch of the graph of $y = \sqrt{x}$ (**Figure 1.26**).

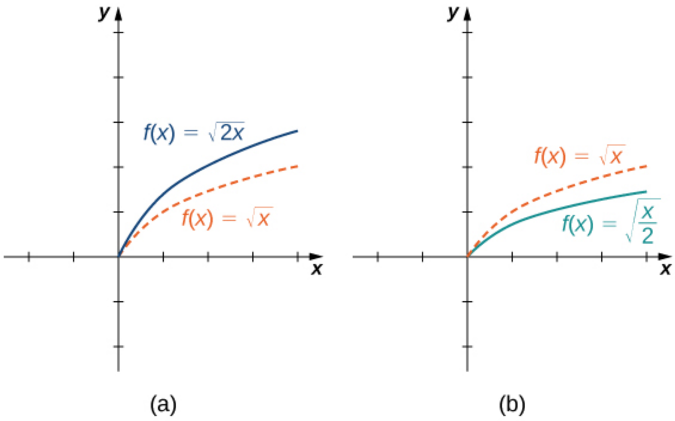


Figure 1.26 (a) If $c > 1$, the graph of $y = f(cx)$ is a horizontal compression of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = f(cx)$ is a horizontal stretch of the graph of $y = f(x)$.

We have explored what happens to the graph of a function f when we multiply f by a constant $c > 0$ to get a new function $cf(x)$. We have also discussed what happens to the graph of a function f when we multiply the independent variable x by $c > 0$ to get a new function $f(cx)$. However, we have not addressed what happens to the graph of the function if the constant c is negative. If we have a constant $c < 0$, we can write c as a positive number multiplied by -1; but, what kind of transformation do we get when we multiply the function or its argument by -1? When we multiply all the outputs by -1, we get a reflection about the x-axis. When we multiply all inputs by -1, we get a reflection about the y-axis. For example, the graph of $f(x) = -(x^3 + 1)$ is the graph of $y = (x^3 + 1)$ reflected about the x-axis. The graph of $f(x) = (-x)^3 + 1$ is the graph of $y = x^3 + 1$ reflected about the y-axis (**Figure 1.27**).

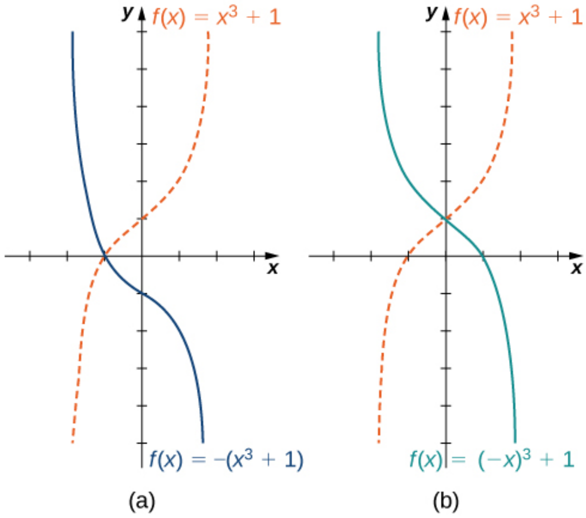


Figure 1.27 (a) The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x-axis. (b) The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y-axis.

If the graph of a function consists of more than one transformation of another graph, it is important to transform the graph in the correct order. Given a function $f(x)$, the graph of the related function $y = cf(a(x + b)) + d$ can be obtained from the graph of $y = f(x)$ by performing the transformations in the following order.

1. Horizontal shift of the graph of $y = f(x)$. If $b > 0$, shift left. If $b < 0$, shift right.
2. Horizontal scaling of the graph of $y = f(x + b)$ by a factor of $|a|$. If $a < 0$, reflect the graph about the y-axis.
3. Vertical scaling of the graph of $y = f(a(x + b))$ by a factor of $|c|$. If $c < 0$, reflect the graph about the x-axis.
4. Vertical shift of the graph of $y = cf(a(x + b))$. If $d > 0$, shift up. If $d < 0$, shift down.

We can summarize the different transformations and their related effects on the graph of a function in the following table.

Transformation of $f(c > 0)$	**Effect on the graph of f**
$f(x) + c$	Vertical shift up c units
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if $0 < c < 1$
$f(cx)$	Horizontal stretch if $0 < c < 1$; horizontal compression if $c > 1$
$-f(x)$	Reflection about the x-axis
$f(-x)$	Reflection about the y-axis

Table 1.7 Transformations of Functions

Example 1.21

Transforming a Function

For each of the following functions, a. and b., sketch a graph by using a sequence of transformations of a well-known function.

a. $f(x) = -|x + 2| - 3$

b. $f(x) = 3\sqrt{-x} + 1$

Solution

a. Starting with the graph of $y = |x|$, shift 2 units to the left, reflect about the x-axis, and then shift down 3 units.

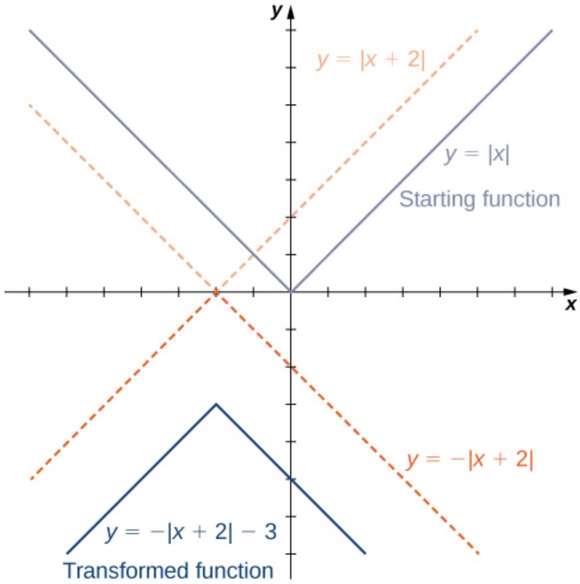


Figure 1.28 The function $f(x) = -|x + 2| - 3$ can be viewed as a sequence of three transformations of the function $y = |x|$.

b. Starting with the graph of $y = \sqrt{x}$, reflect about the y-axis, stretch the graph vertically by a factor of 3, and move up 1 unit.

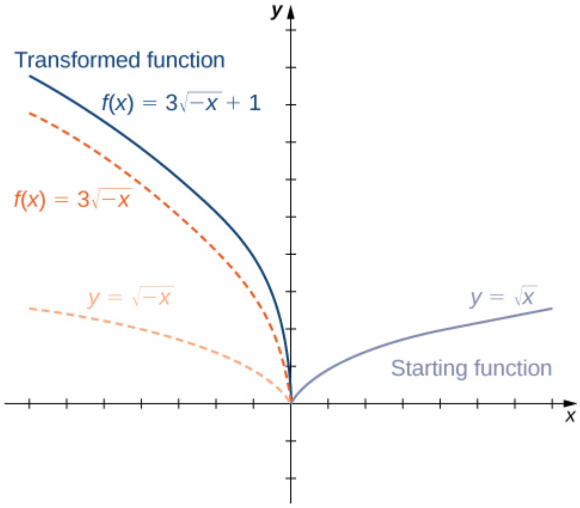


Figure 1.29 The function $f(x) = 3\sqrt{-x} + 1$ can be viewed as a sequence of three transformations of the function $y = \sqrt{x}$.

1.16 Describe how the function $f(x) = -(x + 1)^2 - 4$ can be graphed using the graph of $y = x^2$ and a sequence of transformations.

1.2 EXERCISES

For the following exercises, for each pair of points, a. find the slope of the line passing through the points and b. indicate whether the line is increasing, decreasing, horizontal, or vertical.

59. $(-2, 4)$ and $(1, 1)$

60. $(-1, 4)$ and $(3, -1)$

61. $(3, 5)$ and $(-1, 2)$

62. $(6, 4)$ and $(4, -3)$

63. $(2, 3)$ and $(5, 7)$

64. $(1, 9)$ and $(-8, 5)$

65. $(2, 4)$ and $(1, 4)$

66. $(1, 4)$ and $(1, 0)$

For the following exercises, write the equation of the line satisfying the given conditions in slope-intercept form.

67. Slope $= -6$, passes through $(1, 3)$

68. Slope $= 3$, passes through $(-3, 2)$

69. Slope $= \frac{1}{3}$, passes through $(0, 4)$

70. Slope $= \frac{2}{5}$, x-intercept $= 8$

71. Passing through $(2, 1)$ and $(-2, -1)$

72. Passing through $(-3, 7)$ and $(1, 2)$

73. x-intercept $= 5$ and y-intercept $= -3$

74. x-intercept $= -6$ and y-intercept $= 9$

For the following exercises, for each linear equation, a. give the slope m and y-intercept *b*, if any, and b. graph the line.

75. $y = 2x - 3$

76. $y = -\frac{1}{7}x + 1$

77. $f(x) = -6x$

78. $f(x) = -5x + 4$

79. $4y + 24 = 0$

80. $8x - 4 = 0$

81. $2x + 3y = 6$

82. $6x - 5y + 15 = 0$

For the following exercises, for each polynomial, a. find the degree; b. find the zeros, if any; c. find the y-intercept(s), if any; d. use the leading coefficient to determine the graph's end behavior; and e. determine algebraically whether the polynomial is even, odd, or neither.

83. $f(x) = 2x^2 - 3x - 5$

84. $f(x) = -3x^2 + 6x$

85. $f(x) = \frac{1}{2}x^2 - 1$

86. $f(x) = x^3 + 3x^2 - x - 3$

87. $f(x) = 3x - x^3$

For the following exercises, use the graph of $f(x) = x^2$ to graph each transformed function g.

88. $g(x) = x^2 - 1$

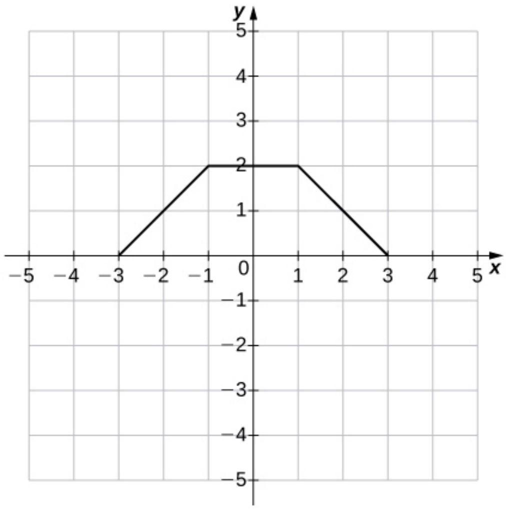
89. $g(x) = (x + 3)^2 + 1$

For the following exercises, use the graph of $f(x) = \sqrt{x}$ to graph each transformed function g.

90. $g(x) = \sqrt{x + 2}$

91. $g(x) = -\sqrt{x} - 1$

For the following exercises, use the graph of $y = f(x)$ to graph each transformed function g.



92. $g(x) = f(x) + 1$

93. $g(x) = f(x - 1) + 2$

For the following exercises, for each of the piecewise-defined functions, a. evaluate at the given values of the independent variable and b. sketch the graph.

94. $f(x) = \begin{cases} 4x + 3, & x \leq 0 \\ -x + 1, & x > 0 \end{cases}$; $f(-3)$; $f(0)$; $f(2)$

95. $f(x) = \begin{cases} x^2 - 3, & x < 0 \\ 4x - 3, & x \geq 0 \end{cases}$; $f(-4)$; $f(0)$; $f(2)$

96. $h(x) = \begin{cases} x + 1, & x \leq 5 \\ 4, & x > 5 \end{cases}$; $h(0)$; $h(\pi)$; $h(5)$

97. $g(x) = \begin{cases} \dfrac{3}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$; $g(0)$; $g(-4)$; $g(2)$

For the following exercises, determine whether the statement is *true or false*. Explain why.

98. $f(x) = (4x + 1)/(7x - 2)$ is a transcendental function.

99. $g(x) = \sqrt[3]{x}$ is an odd root function

100. A logarithmic function is an algebraic function.

101. A function of the form $f(x) = x^b$, where b is a real valued constant, is an exponential function.

102. The domain of an even root function is all real numbers.

103. **[T]** A company purchases some computer equipment for \$20,500. At the end of a 3-year period, the value of the equipment has decreased linearly to \$12,300.

a. Find a function $y = V(t)$ that determines the value V of the equipment at the end of t years.
b. Find and interpret the meaning of the x- and y-intercepts for this situation.
c. What is the value of the equipment at the end of 5 years?
d. When will the value of the equipment be \$3000?

104. **[T]** Total online shopping during the Christmas holidays has increased dramatically during the past 5 years. In 2012 $(t = 0)$, total online holiday sales were \$42.3 billion, whereas in 2013 they were \$48.1 billion.

a. Find a linear function S that estimates the total online holiday sales in the year t.
b. Interpret the slope of the graph of S.
c. Use part a. to predict the year when online shopping during Christmas will reach \$60 billion.

105. **[T]** A family bakery makes cupcakes and sells them at local outdoor festivals. For a music festival, there is a fixed cost of \$125 to set up a cupcake stand. The owner estimates that it costs \$0.75 to make each cupcake. The owner is interested in determining the total cost C as a function of number of cupcakes made.

a. Find a linear function that relates cost C to x, the number of cupcakes made.
b. Find the cost to bake 160 cupcakes.
c. If the owner sells the cupcakes for \$1.50 apiece, how many cupcakes does she need to sell to start making profit? (*Hint*: Use the INTERSECTION function on a calculator to find this number.)

106. **[T]** A house purchased for \$250,000 is expected to be worth twice its purchase price in 18 years.

a. Find a linear function that models the price P of the house versus the number of years t since the original purchase.
b. Interpret the slope of the graph of P.
c. Find the price of the house 15 years from when it was originally purchased.

107. **[T]** A car was purchased for \$26,000. The value of the car depreciates by \$1500 per year.

a. Find a linear function that models the value V of the car after t years.
b. Find and interpret $V(4)$.

108. **[T]** A condominium in an upscale part of the city was purchased for \$432,000. In 35 years it is worth \$60,500. Find the rate of depreciation.

109. **[T]** The total cost C (in thousands of dollars) to produce a certain item is modeled by the function $C(x) = 10.50x + 28{,}500$, where x is the number of items produced. Determine the cost to produce 175 items.

110. **[T]** A professor asks her class to report the amount of time t they spent writing two assignments. Most students report that it takes them about 45 minutes to type a four-page assignment and about 1.5 hours to type a nine-page assignment.

a. Find the linear function $y = N(t)$ that models this situation, where N is the number of pages typed and t is the time in minutes.
b. Use part a. to determine how many pages can be typed in 2 hours.
c. Use part a. to determine how long it takes to type a 20-page assignment.

111. **[T]** The output (as a percent of total capacity) of nuclear power plants in the United States can be modeled by the function $P(t) = 1.8576t + 68.052,$ where t is time in years and $t = 0$ corresponds to the beginning of 2000. Use the model to predict the percentage output in 2015.

112. **[T]** The admissions office at a public university estimates that 65% of the students offered admission to the class of 2019 will actually enroll.

a. Find the linear function $y = N(x),$ where N is the number of students that actually enroll and x is the number of all students offered admission to the class of 2019.
b. If the university wants the 2019 freshman class size to be 1350, determine how many students should be admitted.

1.3 | Trigonometric Functions

Learning Objectives

1.3.1 Convert angle measures between degrees and radians.
1.3.2 Recognize the triangular and circular definitions of the basic trigonometric functions.
1.3.3 Write the basic trigonometric identities.
1.3.4 Identify the graphs and periods of the trigonometric functions.
1.3.5 Describe the shift of a sine or cosine graph from the equation of the function.

Trigonometric functions are used to model many phenomena, including sound waves, vibrations of strings, alternating electrical current, and the motion of pendulums. In fact, almost any repetitive, or cyclical, motion can be modeled by some combination of trigonometric functions. In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

Radian Measure

To use trigonometric functions, we first must understand how to measure the angles. Although we can use both radians and degrees, **radians** are a more natural measurement because they are related directly to the unit circle, a circle with radius 1. The radian measure of an angle is defined as follows. Given an angle $\theta,$ let s be the length of the corresponding arc on the unit circle (Figure 1.30). We say the angle corresponding to the arc of length 1 has radian measure 1.

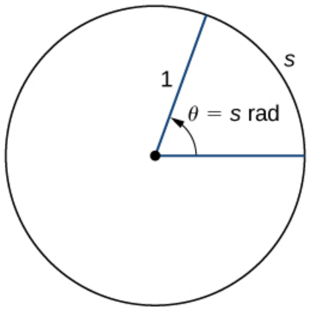


Figure 1.30 The radian measure of an angle θ is the arc length s of the associated arc on the unit circle.

Since an angle of $360°$ corresponds to the circumference of a circle, or an arc of length $2\pi,$ we conclude that an angle with a degree measure of $360°$ has a radian measure of $2\pi.$ Similarly, we see that $180°$ is equivalent to π radians. Table 1.8 shows the relationship between common degree and radian values.

Degrees	Radians	Degrees	Radians
0	0	120	$2\pi/3$
30	$\pi/6$	135	$3\pi/4$
45	$\pi/4$	150	$5\pi/6$
60	$\pi/3$	180	π
90	$\pi/2$		

Table 1.8 Common Angles Expressed in Degrees and Radians

Example 1.22

Converting between Radians and Degrees

a. Express $225°$ using radians.

b. Express $5\pi/3$ rad using degrees.

Solution

Use the fact that $180°$ is equivalent to π radians as a conversion factor: $1 = \frac{\pi \text{ rad}}{180°} = \frac{180°}{\pi \text{ rad}}$.

a. $225° = 225° \cdot \frac{\pi}{180°} = \frac{5\pi}{4}$ rad

b. $\frac{5\pi}{3}$ rad $= \frac{5\pi}{3} \cdot \frac{180°}{\pi} = 300°$

1.17 Express $210°$ using radians. Express $11\pi/6$ rad using degrees.

The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angle measures, in radians or degrees, to find the coordinates of a point on any circle—not only on a unit circle—or to find an angle given a point on a circle. They also define the relationship among the sides and angles of a triangle.

To define the trigonometric functions, first consider the unit circle centered at the origin and a point $P = (x, y)$ on the unit circle. Let θ be an angle with an initial side that lies along the positive x-axis and with a terminal side that is the line segment OP. An angle in this position is said to be in *standard position* (Figure 1.31). We can then define the values of the six trigonometric functions for θ in terms of the coordinates x and y.

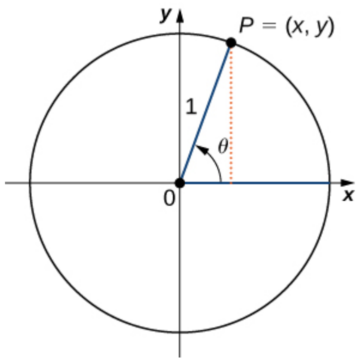


Figure 1.31 The angle θ is in standard position. The values of the trigonometric functions for θ are defined in terms of the coordinates x and y.

Definition

Let $P = (x, y)$ be a point on the unit circle centered at the origin O. Let θ be an angle with an initial side along the positive x-axis and a terminal side given by the line segment OP. The **trigonometric functions** are then defined as

$$\begin{aligned} \sin\theta &= y & \csc\theta &= \frac{1}{y} \\ \cos\theta &= x & \sec\theta &= \frac{1}{x} \\ \tan\theta &= \frac{y}{x} & \cot\theta &= \frac{x}{y} \end{aligned} \tag{1.9}$$

If $x = 0$, $\sec\theta$ and $\tan\theta$ are undefined. If $y = 0$, then $\cot\theta$ and $\csc\theta$ are undefined.

We can see that for a point $P = (x, y)$ on a circle of radius r with a corresponding angle θ, the coordinates x and y satisfy

$$\begin{aligned} \cos\theta &= \frac{x}{r} \\ x &= r\cos\theta \\ \sin\theta &= \frac{y}{r} \\ y &= r\sin\theta. \end{aligned}$$

The values of the other trigonometric functions can be expressed in terms of x, y, and r (**Figure 1.32**).

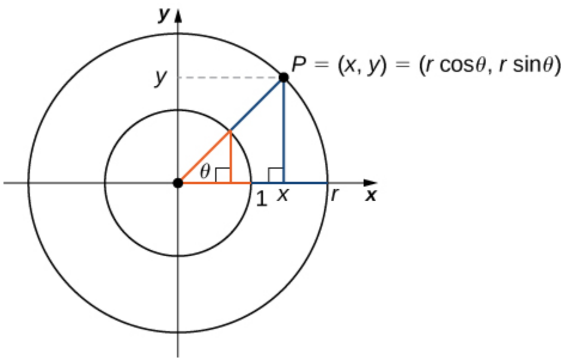


Figure 1.32 For a point $P = (x, y)$ on a circle of radius r, the coordinates x and y satisfy $x = r\cos\theta$ and $y = r\sin\theta$.

Table 1.9 shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of $\sin\theta$ and $\cos\theta$.

$\boldsymbol{\theta}$	$\boldsymbol{\sin\theta}$	$\boldsymbol{\cos\theta}$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Table 1.9 Values of $\sin\theta$ and $\cos\theta$ at Major Angles θ in the First Quadrant

Example 1.23

Evaluating Trigonometric Functions

Evaluate each of the following expressions.

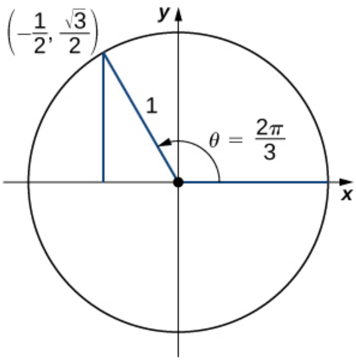
a. $\sin\left(\frac{2\pi}{3}\right)$

b. $\cos\left(-\frac{5\pi}{6}\right)$

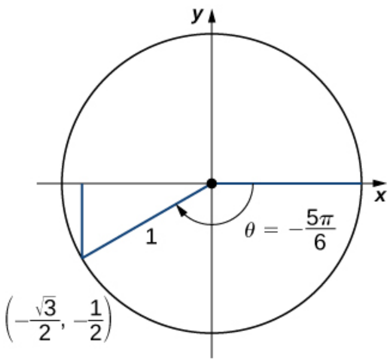
c. $\tan\left(\frac{15\pi}{4}\right)$

Solution

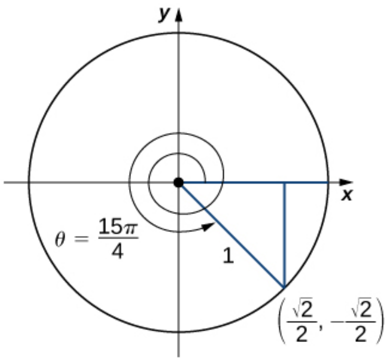
a. On the unit circle, the angle $\theta = \frac{2\pi}{3}$ corresponds to the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore, $\sin\left(\frac{2\pi}{3}\right) = y = \frac{\sqrt{3}}{2}$.



b. An angle $\theta = -\frac{5\pi}{6}$ corresponds to a revolution in the negative direction, as shown. Therefore, $\cos\left(-\frac{5\pi}{6}\right) = x = -\frac{\sqrt{3}}{2}$.



c. An angle $\theta = \frac{15\pi}{4} = 2\pi + \frac{7\pi}{4}$. Therefore, this angle corresponds to more than one revolution, as shown. Knowing the fact that an angle of $\frac{7\pi}{4}$ corresponds to the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, we can conclude that $\tan\left(\frac{15\pi}{4}\right) = \frac{y}{x} = -1$.



1.18 Evaluate $\cos(3\pi/4)$ and $\sin(-\pi/6)$.

As mentioned earlier, the ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle. Let θ be one of the acute angles. Let A be the length of the adjacent leg, O be the length of the opposite leg, and H be the length of the hypotenuse. By inscribing the triangle into a circle of radius H, as shown in **Figure 1.33**, we see that $A, H,$ and O satisfy the following relationships with θ:

$$\begin{aligned} \sin\theta &= \frac{O}{H} & \csc\theta &= \frac{H}{O} \\ \cos\theta &= \frac{A}{H} & \sec\theta &= \frac{H}{A} \\ \tan\theta &= \frac{O}{A} & \cot\theta &= \frac{A}{O} \end{aligned}$$

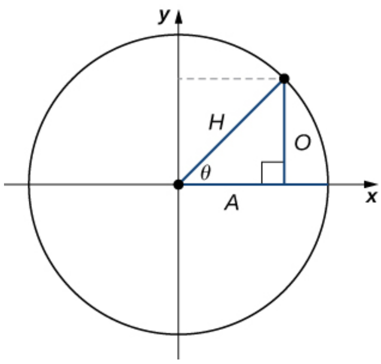


Figure 1.33 By inscribing a right triangle in a circle, we can express the ratios of the side lengths in terms of the trigonometric functions evaluated at θ.

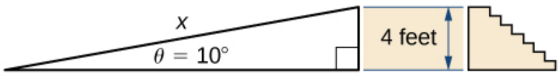
Example 1.24

Constructing a Wooden Ramp

A wooden ramp is to be built with one end on the ground and the other end at the top of a short staircase. If the top of the staircase is 4 ft from the ground and the angle between the ground and the ramp is to be $10°$, how long does the ramp need to be?

Solution

Let x denote the length of the ramp. In the following image, we see that x needs to satisfy the equation $\sin(10°) = 4/x$. Solving this equation for x, we see that $x = 4/\sin(10°) \approx 23.035$ ft.



1.19 A house painter wants to lean a 20-ft ladder against a house. If the angle between the base of the ladder and the ground is to be $60°$, how far from the house should she place the base of the ladder?

Trigonometric Identities

A **trigonometric identity** is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined. We can use the identities to help us solve or simplify equations. The main trigonometric identities are listed next.

Rule: Trigonometric Identities

Reciprocal identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

Addition and subtraction formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Double-angle formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

Example 1.25

Solving Trigonometric Equations

For each of the following equations, use a trigonometric identity to find all solutions.

a. $1 + \cos(2\theta) = \cos\theta$

b. $\sin(2\theta) = \tan\theta$

Solution

a. Using the double-angle formula for $\cos(2\theta)$, we see that θ is a solution of

$$1 + \cos(2\theta) = \cos\theta$$

if and only if

$$1 + 2\cos^2\theta - 1 = \cos\theta,$$

which is true if and only if

$$2\cos^2\theta - \cos\theta = 0.$$

To solve this equation, it is important to note that we need to factor the left-hand side and not divide both sides of the equation by $\cos\theta$. The problem with dividing by $\cos\theta$ is that it is possible that $\cos\theta$ is zero. In fact, if we did divide both sides of the equation by $\cos\theta$, we would miss some of the solutions of the original equation. Factoring the left-hand side of the equation, we see that θ is a solution of this equation if and only if

$$\cos\theta(2\cos\theta - 1) = 0.$$

Since $\cos\theta = 0$ when

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \ldots,$$

and $\cos\theta = 1/2$ when

$$\theta = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi, \ldots \text{ or } \theta = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi, \ldots,$$

we conclude that the set of solutions to this equation is

$$\theta = \frac{\pi}{2} + n\pi, \theta = \frac{\pi}{3} + 2n\pi, \text{ and } \theta = -\frac{\pi}{3} + 2n\pi, n = 0, \pm 1, \pm 2, \ldots.$$

b. Using the double-angle formula for $\sin(2\theta)$ and the reciprocal identity for $\tan(\theta)$, the equation can be written as

$$2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}.$$

To solve this equation, we multiply both sides by $\cos\theta$ to eliminate the denominator, and say that if θ satisfies this equation, then θ satisfies the equation

$$2\sin\theta\cos^2\theta - \sin\theta = 0.$$

However, we need to be a little careful here. Even if θ satisfies this new equation, it may not satisfy the original equation because, to satisfy the original equation, we would need to be able to divide both sides of the equation by $\cos\theta$. However, if $\cos\theta = 0$, we cannot divide both sides of the equation by $\cos\theta$. Therefore, it is possible that we may arrive at extraneous solutions. So, at the end, it is important to check for extraneous solutions. Returning to the equation, it is important that we factor $\sin\theta$ out of both terms on the left-hand side instead of dividing both sides of the equation by $\sin\theta$. Factoring the left-hand side of the equation, we can rewrite this equation as

$$\sin\theta(2\cos^2\theta - 1) = 0.$$

Therefore, the solutions are given by the angles θ such that $\sin\theta = 0$ or $\cos^2\theta = 1/2$. The solutions of the first equation are $\theta = 0, \pm\pi, \pm 2\pi, \ldots$. The solutions of the second equation are $\theta = \pi/4, (\pi/4) \pm (\pi/2), (\pi/4) \pm \pi, \ldots$. After checking for extraneous solutions, the set of solutions to the equation is

$$\theta = n\pi \quad \text{and} \quad \theta = \frac{\pi}{4} + \frac{n\pi}{2},\ n = 0, \pm 1, \pm 2, \ldots.$$

1.20 Find all solutions to the equation $\cos(2\theta) = \sin\theta$.

Example 1.26

Proving a Trigonometric Identity

Prove the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$.

Solution

We start with the identity

$$\sin^2\theta + \cos^2\theta = 1.$$

Dividing both sides of this equation by $\cos^2\theta$, we obtain

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}.$$

Since $\sin\theta/\cos\theta = \tan\theta$ and $1/\cos\theta = \sec\theta$, we conclude that

$$\tan^2\theta + 1 = \sec^2\theta.$$

1.21 Prove the trigonometric identity $1 + \cot^2\theta = \csc^2\theta$.

Graphs and Periods of the Trigonometric Functions

We have seen that as we travel around the unit circle, the values of the trigonometric functions repeat. We can see this pattern in the graphs of the functions. Let $P = (x, y)$ be a point on the unit circle and let θ be the corresponding angle . Since the angle θ and $\theta + 2\pi$ correspond to the same point P, the values of the trigonometric functions at θ and at $\theta + 2\pi$ are the same. Consequently, the trigonometric functions are **periodic functions.** The period of a function f is defined to be the smallest positive value p such that $f(x + p) = f(x)$ for all values x in the domain of f. The sine, cosine, secant, and cosecant functions have a period of 2π. Since the tangent and cotangent functions repeat on an interval of length π, their period is π (**Figure 1.34**).

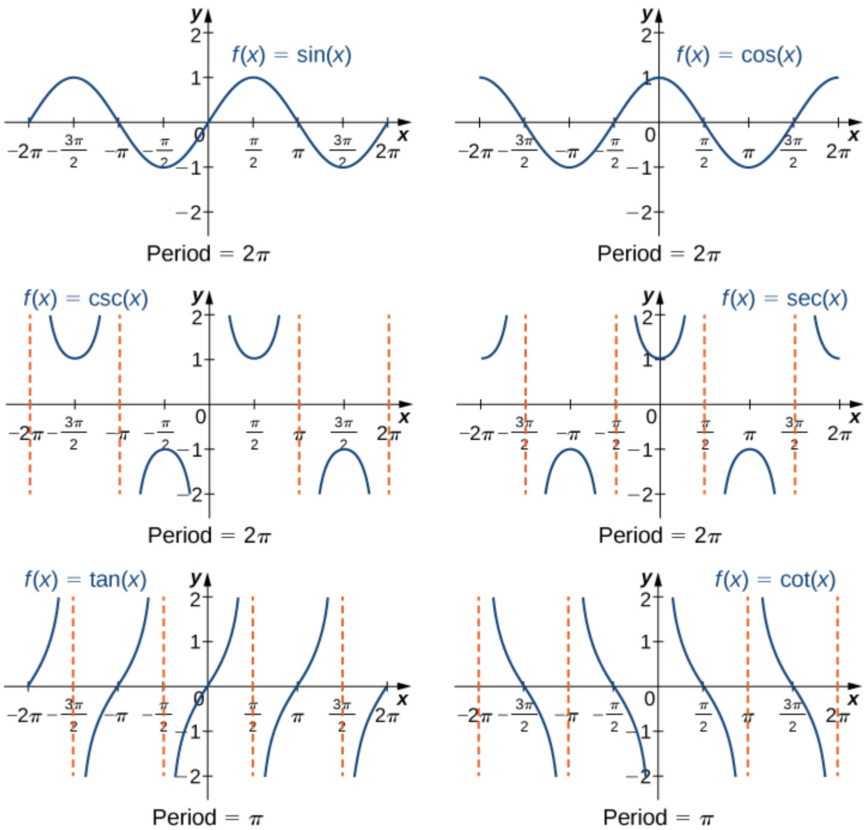


Figure 1.34 The six trigonometric functions are periodic.

Just as with algebraic functions, we can apply transformations to trigonometric functions. In particular, consider the following function:

$$f(x) = A\sin(B(x-\alpha)) + C. \tag{1.10}$$

In **Figure 1.35**, the constant α causes a horizontal or phase shift. The factor B changes the period. This transformed sine function will have a period $2\pi/|B|$. The factor A results in a vertical stretch by a factor of $|A|$. We say $|A|$ is the "amplitude of f." The constant C causes a vertical shift.

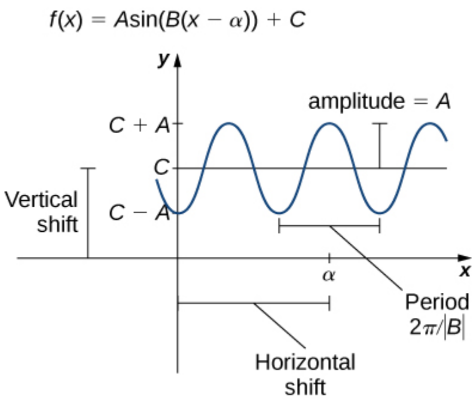


Figure 1.35 A graph of a general sine function.

Notice in **Figure 1.34** that the graph of $y = \cos x$ is the graph of $y = \sin x$ shifted to the left $\pi/2$ units. Therefore, we can write $\cos x = \sin(x + \pi/2)$. Similarly, we can view the graph of $y = \sin x$ as the graph of $y = \cos x$ shifted right $\pi/2$ units, and state that $\sin x = \cos(x - \pi/2)$.

A shifted sine curve arises naturally when graphing the number of hours of daylight in a given location as a function of the day of the year. For example, suppose a city reports that June 21 is the longest day of the year with 15.7 hours and December 21 is the shortest day of the year with 8.3 hours. It can be shown that the function

$$h(t) = 3.7\sin\left(\frac{2\pi}{365}(t - 80.5)\right) + 12$$

is a model for the number of hours of daylight h as a function of day of the year t (**Figure 1.36**).

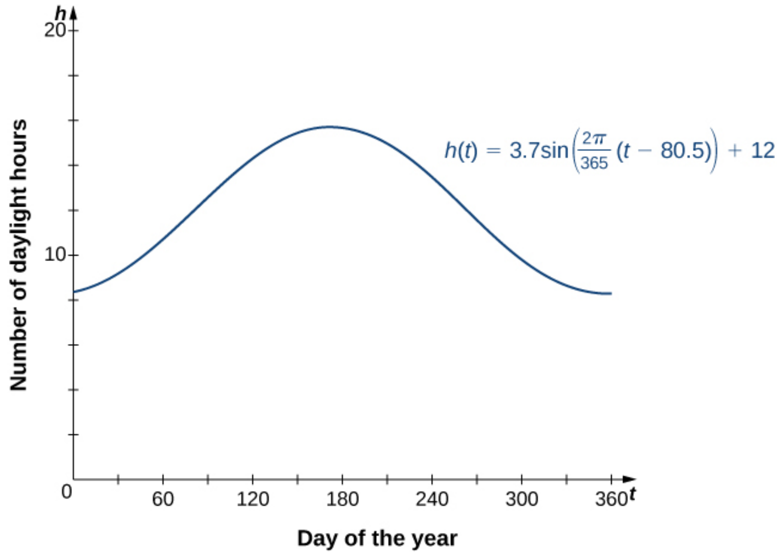


Figure 1.36 The hours of daylight as a function of day of the year can be modeled by a shifted sine curve.

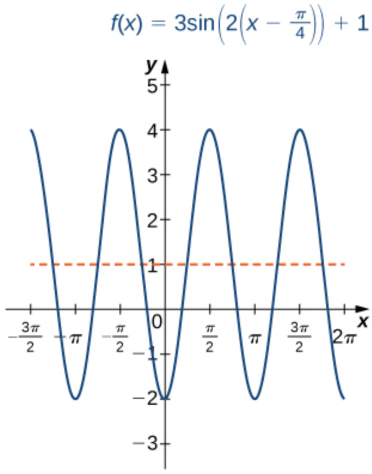
Example 1.27

Sketching the Graph of a Transformed Sine Curve

Sketch a graph of $f(x) = 3\sin\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$.

Solution

This graph is a phase shift of $y = \sin(x)$ to the right by $\pi/4$ units, followed by a horizontal compression by a factor of 2, a vertical stretch by a factor of 3, and then a vertical shift by 1 unit. The period of f is π.



1.22 Describe the relationship between the graph of $f(x) = 3\sin(4x) - 5$ and the graph of $y = \sin(x)$.

1.3 EXERCISES

For the following exercises, convert each angle in degrees to radians. Write the answer as a multiple of π.

113. $240°$

114. $15°$

115. $-60°$

116. $-225°$

117. $330°$

For the following exercises, convert each angle in radians to degrees.

118. $\frac{\pi}{2}$ rad

119. $\frac{7\pi}{6}$ rad

120. $\frac{11\pi}{2}$ rad

121. -3π rad

122. $\frac{5\pi}{12}$ rad

Evaluate the following functional values.

123. $\cos\left(\frac{4\pi}{3}\right)$

124. $\tan\left(\frac{19\pi}{4}\right)$

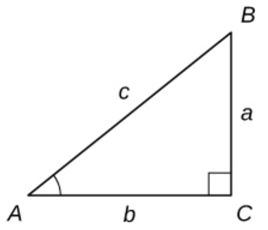
125. $\sin\left(-\frac{3\pi}{4}\right)$

126. $\sec\left(\frac{\pi}{6}\right)$

127. $\sin\left(\frac{\pi}{12}\right)$

128. $\cos\left(\frac{5\pi}{12}\right)$

For the following exercises, consider triangle *ABC*, a right triangle with a right angle at *C*. a. Find the missing side of the triangle. b. Find the six trigonometric function values for the angle at *A*. Where necessary, round to one decimal place.



129. $a = 4,\ c = 7$

130. $a = 21,\ c = 29$

131. $a = 85.3,\ b = 125.5$

132. $b = 40,\ c = 41$

133. $a = 84,\ b = 13$

134. $b = 28,\ c = 35$

For the following exercises, P is a point on the unit circle. a. Find the (exact) missing coordinate value of each point and b. find the values of the six trigonometric functions for the angle θ with a terminal side that passes through point P. Rationalize denominators.

135. $P\left(\frac{7}{25}, y\right),\ y > 0$

136. $P\left(\frac{-15}{17}, y\right),\ y < 0$

137. $P\left(x, \frac{\sqrt{7}}{3}\right),\ x < 0$

138. $P\left(x, \frac{-\sqrt{15}}{4}\right),\ x > 0$

For the following exercises, simplify each expression by writing it in terms of sines and cosines, then simplify. The final answer does not have to be in terms of sine and cosine only.

139. $\tan^2 x + \sin x \csc x$

140. $\sec x \sin x \cot x$

141. $\frac{\tan^2 x}{\sec^2 x}$

142. $\sec x - \cos x$

143. $(1 + \tan\theta)^2 - 2\tan\theta$

144. $\sin x(\csc x - \sin x)$

145. $\frac{\cos t}{\sin t} + \frac{\sin t}{1 + \cos t}$

146. $\frac{1 + \tan^2 \alpha}{1 + \cot^2 \alpha}$

For the following exercises, verify that each equation is an identity.

147. $\frac{\tan\theta\cot\theta}{\csc\theta} = \sin\theta$

148. $\frac{\sec^2\theta}{\tan\theta} = \sec\theta\csc\theta$

149. $\frac{\sin t}{\csc t} + \frac{\cos t}{\sec t} = 1$

150. $\frac{\sin x}{\cos x + 1} + \frac{\cos x - 1}{\sin x} = 0$

151. $\cot\gamma + \tan\gamma = \sec\gamma\csc\gamma$

152. $\sin^2\beta + \tan^2\beta + \cos^2\beta = \sec^2\beta$

153. $\frac{1}{1 - \sin\alpha} + \frac{1}{1 + \sin\alpha} = 2\sec^2\alpha$

154. $\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \csc^2\theta$

For the following exercises, solve the trigonometric equations on the interval $0 \le \theta < 2\pi$.

155. $2\sin\theta - 1 = 0$

156. $1 + \cos\theta = \frac{1}{2}$

157. $2\tan^2\theta = 2$

158. $4\sin^2\theta - 2 = 0$

159. $\sqrt{3}\cot\theta + 1 = 0$

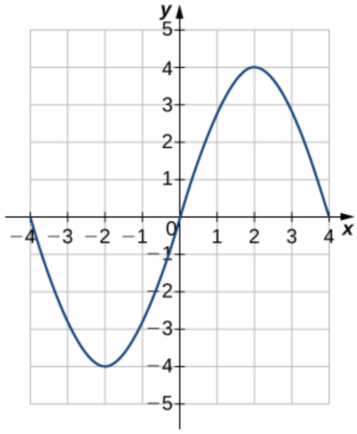
160. $3\sec\theta - 2\sqrt{3} = 0$

161. $2\cos\theta\sin\theta = \sin\theta$

162. $\csc^2\theta + 2\csc\theta + 1 = 0$

For the following exercises, each graph is of the form $y = A\sin Bx$ or $y = A\cos Bx$, where $B > 0$. Write the equation of the graph.

163.



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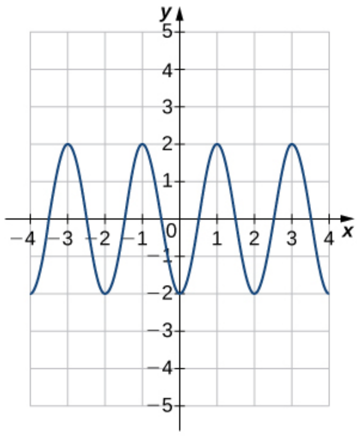

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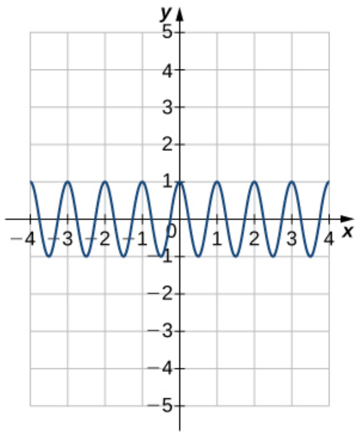

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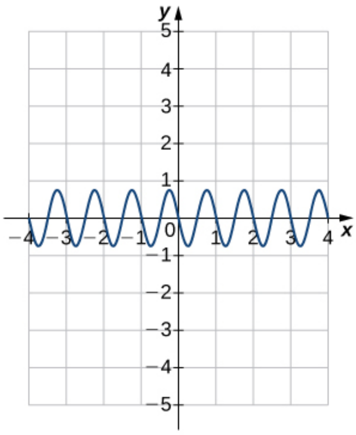


For the following exercises, find a. the amplitude, b. the period, and c. the phase shift with direction for each function.

167. $y = \sin\left(x - \frac{\pi}{4}\right)$

168. $y = 3\cos(2x + 3)$

169. $y = \frac{-1}{2}\sin\left(\frac{1}{4}x\right)$

170. $y = 2\cos\left(x - \frac{\pi}{3}\right)$

171. $y = -3\sin(\pi x + 2)$

172. $y = 4\cos\left(2x - \frac{\pi}{2}\right)$

173. **[T]** The diameter of a wheel rolling on the ground is 40 in. If the wheel rotates through an angle of $120°$, how many inches does it move? Approximate to the nearest whole inch.

174. **[T]** Find the length of the arc intercepted by central angle θ in a circle of radius r. Round to the nearest hundredth. a. $r = 12.8$ cm, $\theta = \frac{5\pi}{6}$ rad b. $r = 4.378$ cm, $\theta = \frac{7\pi}{6}$ rad c. $r = 0.964$ cm, $\theta = 50°$ d. $r = 8.55$ cm, $\theta = 325°$

175. **[T]** As a point P moves around a circle, the measure of the angle changes. The measure of how fast the angle is changing is called *angular speed*, ω, and is given by $\omega = \theta/t$, where θ is in radians and t is time. Find the angular speed for the given data. Round to the nearest thousandth. a. $\theta = \frac{7\pi}{4}$rad, $t = 10$ sec b. $\theta = \frac{3\pi}{5}$rad, $t = 8$ sec c. $\theta = \frac{2\pi}{9}$rad, $t = 1$ min d. $\theta = 23.76$ rad, $t = 14$ min

176. **[T]** A total of 250,000 m^2 of land is needed to build a nuclear power plant. Suppose it is decided that the area on which the power plant is to be built should be circular.
   a. Find the radius of the circular land area.
   b. If the land area is to form a $45°$ sector of a circle instead of a whole circle, find the length of the curved side.

177. **[T]** The area of an isosceles triangle with equal sides of length x is $\frac{1}{2}x^2\sin\theta$, where θ is the angle formed by the two sides. Find the area of an isosceles triangle with equal sides of length 8 in. and angle $\theta = 5\pi/12$ rad.

178. **[T]** A particle travels in a circular path at a constant angular speed ω. The angular speed is modeled by the function $\omega = 9|\cos(\pi t - \pi/12)|$. Determine the angular speed at $t = 9$ sec.

179. **[T]** An alternating current for outlets in a home has voltage given by the function $V(t) = 150\cos 368t$, where V is the voltage in volts at time t in seconds.
   a. Find the period of the function and interpret its meaning.
   b. Determine the number of periods that occur when 1 sec has passed.

180. **[T]** The number of hours of daylight in a northeast city is modeled by the function

$$N(t) = 12 + 3\sin\left[\frac{2\pi}{365}(t - 79)\right],$$

where t is the number of days after January 1.
   a. Find the amplitude and period.
   b. Determine the number of hours of daylight on the longest day of the year.
   c. Determine the number of hours of daylight on the shortest day of the year.
   d. Determine the number of hours of daylight 90 days after January 1.
   e. Sketch the graph of the function for one period starting on January 1.

181. **[T]** Suppose that $T = 50 + 10\sin\left[\frac{\pi}{12}(t - 8)\right]$ is a mathematical model of the temperature (in degrees Fahrenheit) at *t* hours after midnight on a certain day of the week.

a. Determine the amplitude and period.
b. Find the temperature 7 hours after midnight.
c. At what time does $T = 60°$?
d. Sketch the graph of T over $0 \le t \le 24$.

182. **[T]** The function $H(t) = 8\sin\left(\frac{\pi}{6}t\right)$ models the height *H* (in feet) of the tide *t* hours after midnight. Assume that $t = 0$ is midnight.

a. Find the amplitude and period.
b. Graph the function over one period.
c. What is the height of the tide at 4:30 a.m.?

1.4 | Inverse Functions

Learning Objectives

1.4.1 Determine the conditions for when a function has an inverse.
1.4.2 Use the horizontal line test to recognize when a function is one-to-one.
1.4.3 Find the inverse of a given function.
1.4.4 Draw the graph of an inverse function.
1.4.5 Evaluate inverse trigonometric functions.

An inverse function reverses the operation done by a particular function. In other words, whatever a function does, the inverse function undoes it. In this section, we define an inverse function formally and state the necessary conditions for an inverse function to exist. We examine how to find an inverse function and study the relationship between the graph of a function and the graph of its inverse. Then we apply these ideas to define and discuss properties of the inverse trigonometric functions.

Existence of an Inverse Function

We begin with an example. Given a function f and an output $y = f(x)$, we are often interested in finding what value or values x were mapped to y by f. For example, consider the function $f(x) = x^3 + 4$. Since any output $y = x^3 + 4$, we can solve this equation for x to find that the input is $x = \sqrt[3]{y-4}$. This equation defines x as a function of y. Denoting this function as f^{-1}, and writing $x = f^{-1}(y) = \sqrt[3]{y-4}$, we see that for any x in the domain of f, $f^{-1}(f(x)) = f^{-1}\left(x^3 + 4\right) = x$. Thus, this new function, f^{-1}, "undid" what the original function f did. A function with this property is called the inverse function of the original function.

Definition

Given a function f with domain D and range R, its **inverse function** (if it exists) is the function f^{-1} with domain R and range D such that $f^{-1}(y) = x$ if $f(x) = y$. In other words, for a function f and its inverse f^{-1},

$$f^{-1}(f(x)) = x \text{ for all } x \text{ in } D, \text{ and } f\left(f^{-1}(y)\right) = y \text{ for all } y \text{ in } R. \tag{1.11}$$

Note that f^{-1} is read as "f inverse." Here, the -1 is not used as an exponent and $f^{-1}(x) \neq 1/f(x)$. Figure 1.37 shows the relationship between the domain and range of f and the domain and range of f^{-1}.

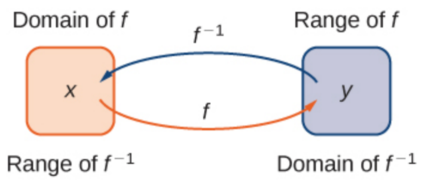


Figure 1.37 Given a function f and its inverse f^{-1}, $f^{-1}(y) = x$ if and only if $f(x) = y$. The range of f becomes the domain of f^{-1} and the domain of f becomes the range of f^{-1}.

Recall that a function has exactly one output for each input. Therefore, to define an inverse function, we need to map each input to exactly one output. For example, let's try to find the inverse function for $f(x) = x^2$. Solving the equation $y = x^2$ for x, we arrive at the equation $x = \pm\sqrt{y}$. This equation does not describe x as a function of y because there are two solutions to this equation for every $y > 0$. The problem with trying to find an inverse function for $f(x) = x^2$ is that two inputs are sent to the same output for each output $y > 0$. The function $f(x) = x^3 + 4$ discussed earlier did not have this problem. For that function, each input was sent to a different output. A function that sends each input to a *different* output is called a one-to-one function.

Definition

We say a f is a **one-to-one function** if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

One way to determine whether a function is one-to-one is by looking at its graph. If a function is one-to-one, then no two inputs can be sent to the same output. Therefore, if we draw a horizontal line anywhere in the xy-plane, according to the **horizontal line test**, it cannot intersect the graph more than once. We note that the horizontal line test is different from the vertical line test. The vertical line test determines whether a graph is the graph of a function. The horizontal line test determines whether a function is one-to-one (Figure 1.38).

Rule: Horizontal Line Test

A function f is one-to-one if and only if every horizontal line intersects the graph of f no more than once.

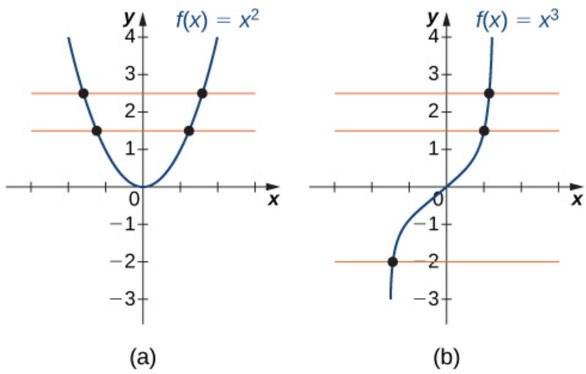
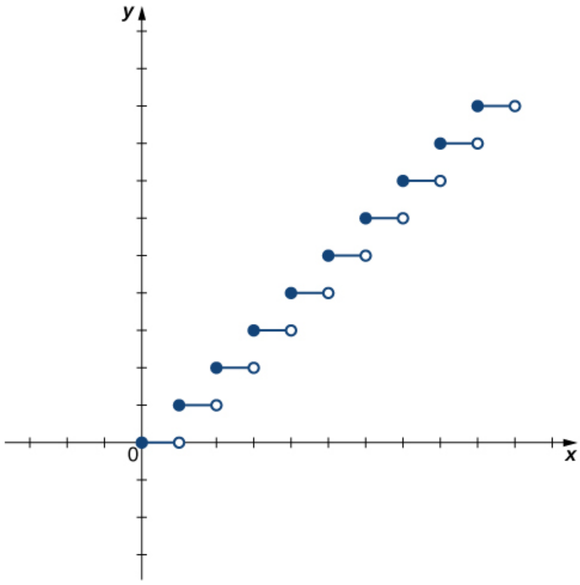


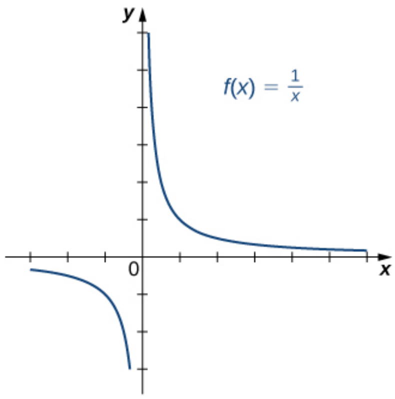
Figure 1.38 (a) The function $f(x) = x^2$ is not one-to-one because it fails the horizontal line test. (b) The function $f(x) = x^3$ is one-to-one because it passes the horizontal line test.

Example 1.28

Determining Whether a Function Is One-to-One

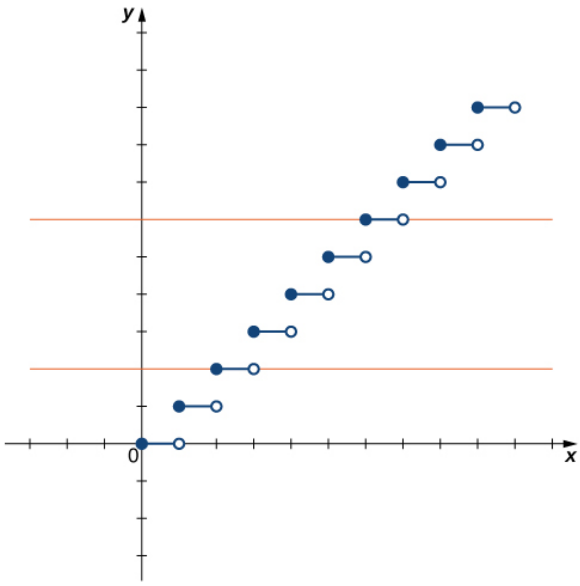
For each of the following functions, use the horizontal line test to determine whether it is one-to-one.

a. 

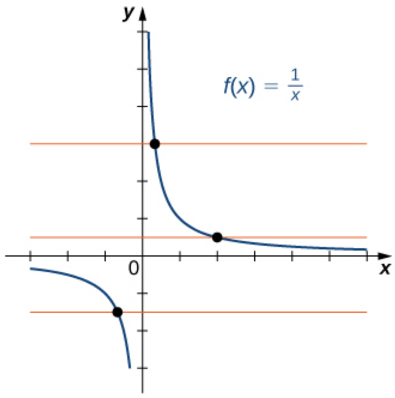
b. 

Solution

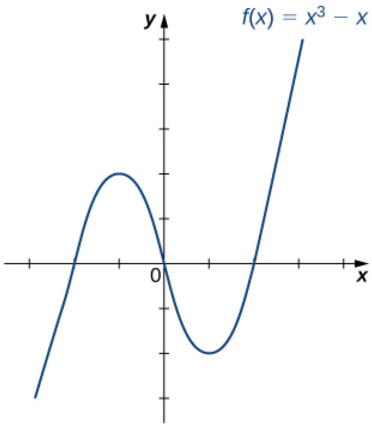
a. Since the horizontal line $y = n$ for any integer $n \geq 0$ intersects the graph more than once, this function is not one-to-one.



b. Since every horizontal line intersects the graph once (at most), this function is one-to-one.



1.23 Is the function f graphed in the following image one-to-one?



Finding a Function's Inverse

We can now consider one-to-one functions and show how to find their inverses. Recall that a function maps elements in the domain of f to elements in the range of f. The inverse function maps each element from the range of f back to its corresponding element from the domain of f. Therefore, to find the inverse function of a one-to-one function f, given any y in the range of f, we need to determine which x in the domain of f satisfies $f(x) = y$. Since f is one-to-one, there is exactly one such value x. We can find that value x by solving the equation $f(x) = y$ for x. Doing so, we are able to write x as a function of y where the domain of this function is the range of f and the range of this new function is the domain of f. Consequently, this function is the inverse of f, and we write $x = f^{-1}(y)$. Since we typically use the variable x to denote the independent variable and y to denote the dependent variable, we often interchange the roles of x and y, and write $y = f^{-1}(x)$. Representing the inverse function in this way is also helpful later when we graph a function f and its inverse f^{-1} on the same axes.

Problem-Solving Strategy: Finding an Inverse Function

1. Solve the equation $y = f(x)$ for x.
2. Interchange the variables x and y and write $y = f^{-1}(x)$.

Example 1.29

Finding an Inverse Function

Find the inverse for the function $f(x) = 3x - 4$. State the domain and range of the inverse function. Verify that $f^{-1}(f(x)) = x$.

Solution

Follow the steps outlined in the strategy.

Step 1. If $y = 3x - 4$, then $3x = y + 4$ and $x = \frac{1}{3}y + \frac{4}{3}$.

Step 2. Rewrite as $y = \frac{1}{3}x + \frac{4}{3}$ and let $y = f^{-1}(x)$.

Therefore, $f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$.

Since the domain of f is $(-\infty, \infty)$, the range of f^{-1} is $(-\infty, \infty)$. Since the range of f is $(-\infty, \infty)$, the domain of f^{-1} is $(-\infty, \infty)$.

You can verify that $f^{-1}(f(x)) = x$ by writing

$$f^{-1}(f(x)) = f^{-1}(3x - 4) = \frac{1}{3}(3x - 4) + \frac{4}{3} = x - \frac{4}{3} + \frac{4}{3} = x.$$

Note that for $f^{-1}(x)$ to be the inverse of $f(x)$, both $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for all x in the domain of the inside function.

1.24 Find the inverse of the function $f(x) = 3x/(x - 2)$. State the domain and range of the inverse function.

Graphing Inverse Functions

Let's consider the relationship between the graph of a function f and the graph of its inverse. Consider the graph of f shown in **Figure 1.39** and a point (a, b) on the graph. Since $b = f(a)$, then $f^{-1}(b) = a$. Therefore, when we graph f^{-1}, the point (b, a) is on the graph. As a result, the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

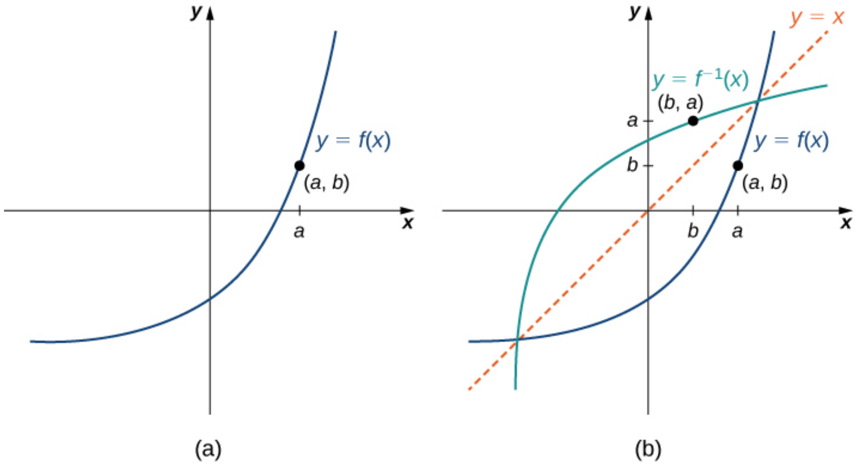
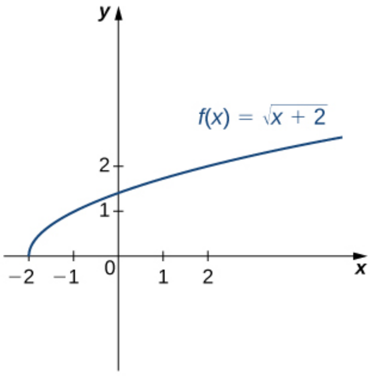


Figure 1.39 (a) The graph of this function f shows point (a, b) on the graph of f. (b) Since (a, b) is on the graph of f, the point (b, a) is on the graph of f^{-1}. The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

Example 1.30

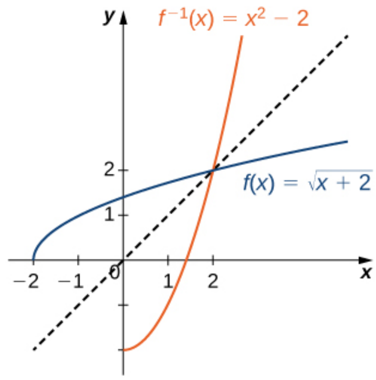
Sketching Graphs of Inverse Functions

For the graph of f in the following image, sketch a graph of f^{-1} by sketching the line $y = x$ and using symmetry. Identify the domain and range of f^{-1}.



Solution

Reflect the graph about the line $y = x$. The domain of f^{-1} is $[0, \infty)$. The range of f^{-1} is $[-2, \infty)$. By using the preceding strategy for finding inverse functions, we can verify that the inverse function is $f^{-1}(x) = x^2 - 2$, as shown in the graph.



1.25 Sketch the graph of $f(x) = 2x + 3$ and the graph of its inverse using the symmetry property of inverse functions.

Restricting Domains

As we have seen, $f(x) = x^2$ does not have an inverse function because it is not one-to-one. However, we can choose a subset of the domain of f such that the function is one-to-one. This subset is called a **restricted domain**. By restricting the domain of f, we can define a new function g such that the domain of g is the restricted domain of f and $g(x) = f(x)$ for all x in the domain of g. Then we can define an inverse function for g on that domain. For example, since $f(x) = x^2$ is one-to-one on the interval $[0, \infty)$, we can define a new function g such that the domain of g is $[0, \infty)$ and $g(x) = x^2$ for all x in its domain. Since g is a one-to-one function, it has an inverse function, given by the formula $g^{-1}(x) = \sqrt{x}$. On the other hand, the function $f(x) = x^2$ is also one-to-one on the domain $(-\infty, 0]$. Therefore, we could also define a new function h such that the domain of h is $(-\infty, 0]$ and $h(x) = x^2$ for all x in the domain of h. Then h is a one-to-one function and must also have an inverse. Its inverse is given by the formula $h^{-1}(x) = -\sqrt{x}$ (**Figure 1.40**).

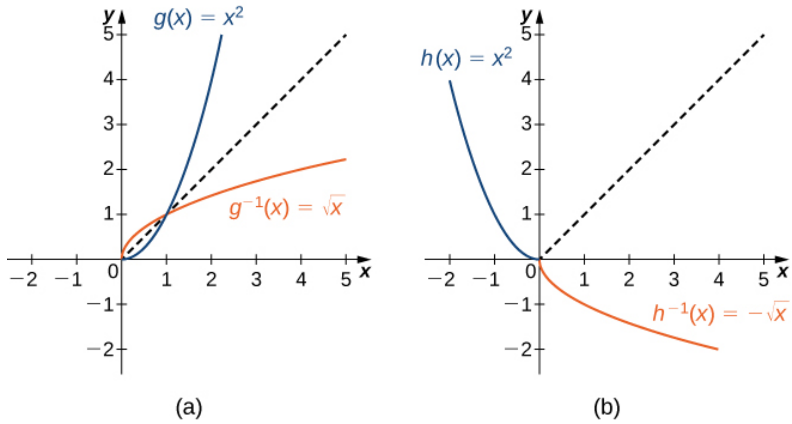


Figure 1.40 (a) For $g(x) = x^2$ restricted to $[0, \infty)$, $g^{-1}(x) = \sqrt{x}$. (b) For $h(x) = x^2$ restricted to $(-\infty, 0]$, $h^{-1}(x) = -\sqrt{x}$.

Example 1.31

Restricting the Domain

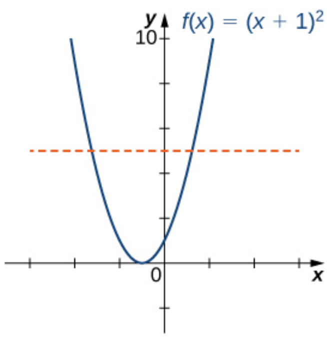
Consider the function $f(x) = (x + 1)^2$.

a. Sketch the graph of f and use the horizontal line test to show that f is not one-to-one.

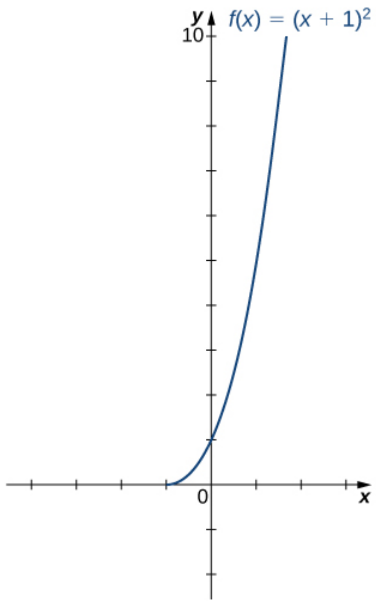
b. Show that f is one-to-one on the restricted domain $[-1, \infty)$. Determine the domain and range for the inverse of f on this restricted domain and find a formula for f^{-1}.

Solution

a. The graph of f is the graph of $y = x^2$ shifted left 1 unit. Since there exists a horizontal line intersecting the graph more than once, f is not one-to-one.



b. On the interval $[-1, \infty)$, f is one-to-one.



The domain and range of f^{-1} are given by the range and domain of f, respectively. Therefore, the domain of f^{-1} is $[0, \infty)$ and the range of f^{-1} is $[-1, \infty)$. To find a formula for f^{-1}, solve the equation $y = (x+1)^2$ for x. If $y = (x+1)^2$, then $x = -1 \pm \sqrt{y}$. Since we are restricting the domain to the interval where $x \geq -1$, we need $\pm\sqrt{y} \geq 0$. Therefore, $x = -1 + \sqrt{y}$. Interchanging x and y, we write $y = -1 + \sqrt{x}$ and conclude that $f^{-1}(x) = -1 + \sqrt{x}$.

1.26 Consider $f(x) = 1/x^2$ restricted to the domain $(-\infty, 0)$. Verify that f is one-to-one on this domain. Determine the domain and range of the inverse of f and find a formula for f^{-1}.

Inverse Trigonometric Functions

The six basic trigonometric functions are periodic, and therefore they are not one-to-one. However, if we restrict the domain of a trigonometric function to an interval where it is one-to-one, we can define its inverse. Consider the sine function (**Figure 1.34**). The sine function is one-to-one on an infinite number of intervals, but the standard convention is to restrict the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. By doing so, we define the inverse sine function on the domain $[-1, 1]$ such that for any x in the interval $[-1, 1]$, the inverse sine function tells us which angle θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ satisfies $\sin\theta = x$. Similarly, we can restrict the domains of the other trigonometric functions to define **inverse trigonometric functions**, which are functions that tell us which angle in a certain interval has a specified trigonometric value.

Definition

The inverse sine function, denoted $\sin^{-1}$ or arcsin, and the inverse cosine function, denoted $\cos^{-1}$ or arccos, are defined on the domain $D = \{x | -1 \leq x \leq 1\}$ as follows:

$$\sin^{-1}(x) = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2};$$
$$\cos^{-1}(x) = y \text{ if and only if } \cos(y) = x \text{ and } 0 \le y \le \pi. \tag{1.12}$$

The inverse tangent function, denoted $\tan^{-1}$ or arctan, and inverse cotangent function, denoted $\cot^{-1}$ or arccot, are defined on the domain $D = \{x| -\infty < x < \infty\}$ as follows:

$$\tan^{-1}(x) = y \text{ if and only if } \tan(y) = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2};$$
$$\cot^{-1}(x) = y \text{ if and only if } \cot(y) = x \text{ and } 0 < y < \pi. \tag{1.13}$$

The inverse cosecant function, denoted $\csc^{-1}$ or arccsc, and inverse secant function, denoted $\sec^{-1}$ or arcsec, are defined on the domain $D = \{x||x| \ge 1\}$ as follows:

$$\csc^{-1}(x) = y \text{ if and only if } \csc(y) = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0;$$
$$\sec^{-1}(x) = y \text{ if and only if } \sec(y) = x \text{ and } 0 \le y \le \pi,\ y \ne \pi/2. \tag{1.14}$$

To graph the inverse trigonometric functions, we use the graphs of the trigonometric functions restricted to the domains defined earlier and reflect the graphs about the line $y = x$ (**Figure 1.41**).

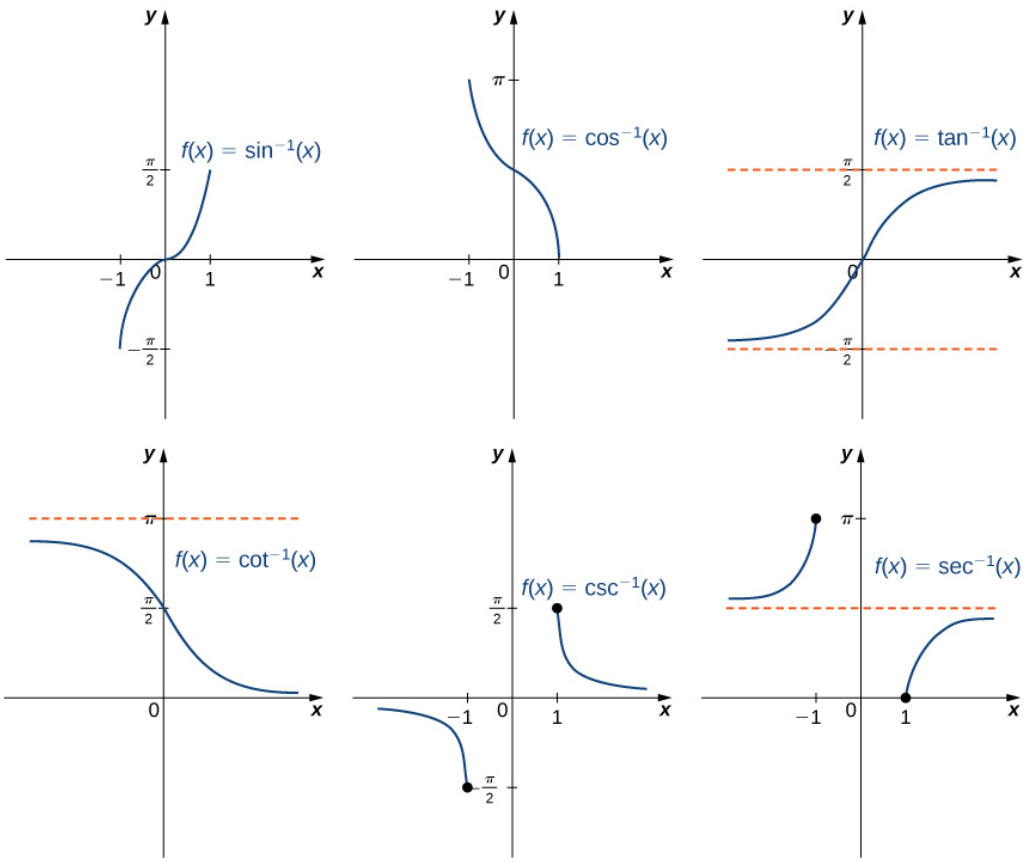


Figure 1.41 The graph of each of the inverse trigonometric functions is a reflection about the line $y = x$ of the corresponding restricted trigonometric function.

Go to the **following site (http://www.openstax.org/l/20_inversefun)** for more comparisons of functions and their inverses.

When evaluating an inverse trigonometric function, the output is an angle. For example, to evaluate $\cos^{-1}\left(\frac{1}{2}\right),$ we need to find an angle θ such that $\cos\theta = \frac{1}{2}.$ Clearly, many angles have this property. However, given the definition of $\cos^{-1},$ we need the angle θ that not only solves this equation, but also lies in the interval $[0, \pi].$ We conclude that $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$

We now consider a composition of a trigonometric function and its inverse. For example, consider the two expressions $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $\sin^{-1}(\sin(\pi)).$ For the first one, we simplify as follows:

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

For the second one, we have

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0.$$

The inverse function is supposed to “undo” the original function, so why isn’t $\sin^{-1}(\sin(\pi)) = \pi$? Recalling our definition of inverse functions, a function f and its inverse f^{-1} satisfy the conditions $f\left(f^{-1}(y)\right) = y$ for all y in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of $f,$ so what happened here? The issue is that the inverse sine function, $\sin^{-1},$ is the inverse of the *restricted* sine function defined on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$ Therefore, for x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$ it is true that $\sin^{-1}(\sin x) = x.$ However, for values of x outside this interval, the equation does not hold, even though $\sin^{-1}(\sin x)$ is defined for all real numbers $x.$

What about $\sin(\sin^{-1} y)$? Does that have a similar issue? The answer is *no*. Since the domain of $\sin^{-1}$ is the interval $[-1, 1],$ we conclude that $\sin(\sin^{-1} y) = y$ if $-1 \le y \le 1$ and the expression is not defined for other values of $y.$ To summarize,

$$\sin(\sin^{-1} y) = y \text{ if } -1 \le y \le 1$$

and

$$\sin^{-1}(\sin x) = x \text{ if } -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Similarly, for the cosine function,

$$\cos(\cos^{-1} y) = y \text{ if } -1 \le y \le 1$$

and

$$\cos^{-1}(\cos x) = x \text{ if } 0 \le x \le \pi.$$

Similar properties hold for the other trigonometric functions and their inverses.

Example 1.32

Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluate each of the following expressions.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

b. $\tan\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$

c. $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

d. $\sin^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

Solution

a. Evaluating $\sin^{-1}\left(-\sqrt{3}/2\right)$ is equivalent to finding the angle θ such that $\sin\theta = -\sqrt{3}/2$ and $-\pi/2 \leq \theta \leq \pi/2.$ The angle $\theta = -\pi/3$ satisfies these two conditions. Therefore, $\sin^{-1}\left(-\sqrt{3}/2\right) = -\pi/3.$

b. First we use the fact that $\tan^{-1}\left(-1/\sqrt{3}\right) = -\pi/6.$ Then $\tan(\pi/6) = -1/\sqrt{3}.$ Therefore, $\tan\left(\tan^{-1}\left(-1/\sqrt{3}\right)\right) = -1/\sqrt{3}.$

c. To evaluate $\cos^{-1}(\cos(5\pi/4)),$ first use the fact that $\cos(5\pi/4) = -\sqrt{2}/2.$ Then we need to find the angle θ such that $\cos(\theta) = -\sqrt{2}/2$ and $0 \leq \theta \leq \pi.$ Since $3\pi/4$ satisfies both these conditions, we have $\cos\left(\cos^{-1}(5\pi/4)\right) = \cos\left(\cos^{-1}\left(-\sqrt{2}/2\right)\right) = 3\pi/4.$

d. Since $\cos(2\pi/3) = -1/2,$ we need to evaluate $\sin^{-1}(-1/2).$ That is, we need to find the angle θ such that $\sin(\theta) = -1/2$ and $-\pi/2 \leq \theta \leq \pi/2.$ Since $-\pi/6$ satisfies both these conditions, we can conclude that $\sin^{-1}(\cos(2\pi/3)) = \sin^{-1}(-1/2) = -\pi/6.$

Student PROJECT

The Maximum Value of a Function

In many areas of science, engineering, and mathematics, it is useful to know the maximum value a function can obtain, even if we don't know its exact value at a given instant. For instance, if we have a function describing the strength of a roof beam, we would want to know the maximum weight the beam can support without breaking. If we have a function that describes the speed of a train, we would want to know its maximum speed before it jumps off the rails. Safe design often depends on knowing maximum values.

This project describes a simple example of a function with a maximum value that depends on two equation coefficients. We will see that maximum values can depend on several factors other than the independent variable *x*.

1. Consider the graph in **Figure 1.42** of the function $y = \sin x + \cos x$. Describe its overall shape. Is it periodic? How do you know?

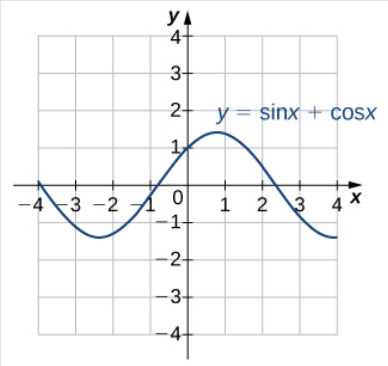


Figure 1.42 The graph of $y = \sin x + \cos x$.

Using a graphing calculator or other graphing device, estimate the x - and y -values of the maximum point for the graph (the first such point where $x > 0$). It may be helpful to express the x -value as a multiple of π.

2. Now consider other graphs of the form $y = A \sin x + B \cos x$ for various values of *A* and *B*. Sketch the graph when $A = 2$ and $B = 1$, and find the x - and *y*-values for the maximum point. (Remember to express the *x*-value as a multiple of π, if possible.) Has it moved?
3. Repeat for $A = 1$, $B = 2$. Is there any relationship to what you found in part (2)?
4. Complete the following table, adding a few choices of your own for *A* and *B*:

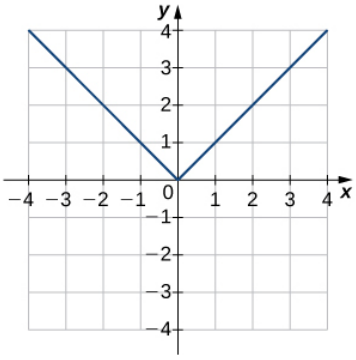
A	*B*	*x*	*y*		*A*	*B*	*x*	*y*
0	1				$\sqrt{3}$	1		
1	0				1	$\sqrt{3}$		
1	1				12	5		
1	2				5	12		
2	1							
2	2							
3	4							
4	3							

5. Try to figure out the formula for the y-values.
6. The formula for the x-values is a little harder. The most helpful points from the table are $(1,\ 1), \left(1,\ \sqrt{3}\right), \left(\sqrt{3},\ 1\right)$. (*Hint*: Consider inverse trigonometric functions.)
7. If you found formulas for parts (5) and (6), show that they work together. That is, substitute the x-value formula you found into $y = A\sin x + B\cos x$ and simplify it to arrive at the y-value formula you found.

1.4 EXERCISES

For the following exercises, use the horizontal line test to determine whether each of the given graphs is one-to-one.

183.



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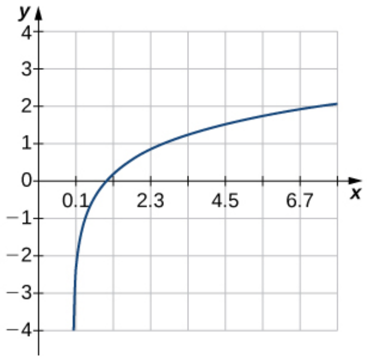

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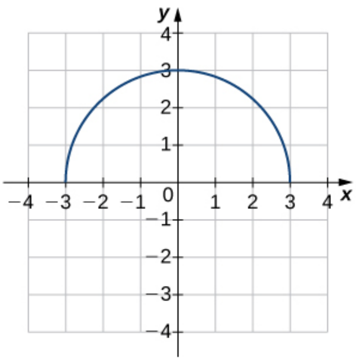

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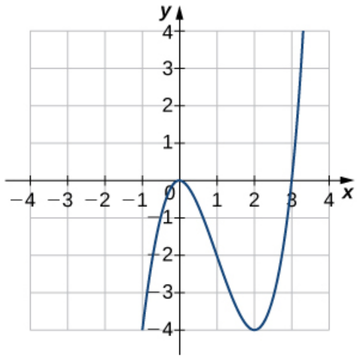

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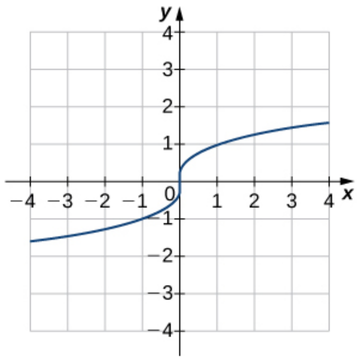

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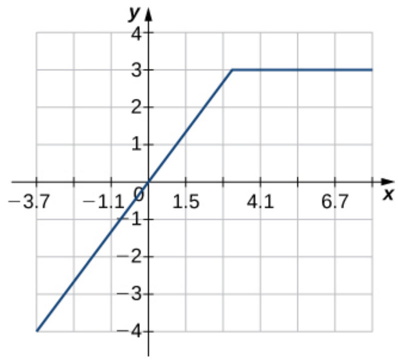


For the following exercises, a. find the inverse function, and b. find the domain and range of the inverse function.

189. $f(x) = x^2 - 4,\ x \geq 0$

190. $f(x) = \sqrt[3]{x - 4}$

191. $f(x) = x^3 + 1$

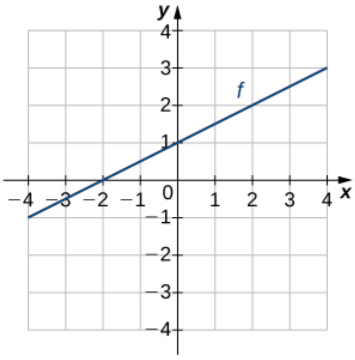
192. $f(x) = (x - 1)^2,\ x \leq 1$

193. $f(x) = \sqrt{x-1}$

194. $f(x) = \dfrac{1}{x+2}$

For the following exercises, use the graph of f to sketch the graph of its inverse function.

195.



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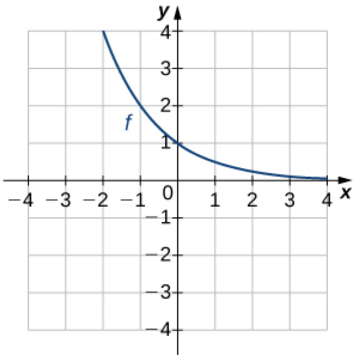

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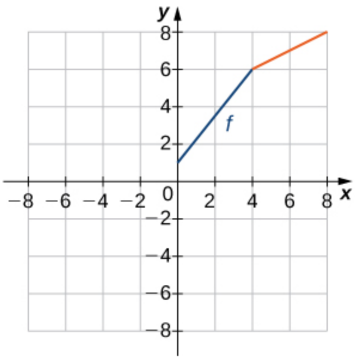
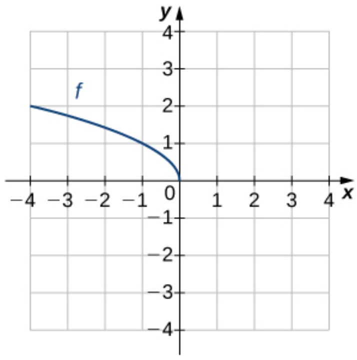


198.



For the following exercises, use composition to determine which pairs of functions are inverses.

199. $f(x) = 8x,\ g(x) = \dfrac{x}{8}$

200. $f(x) = 8x + 3,\ g(x) = \dfrac{x-3}{8}$

201. $f(x) = 5x - 7,\ g(x) = \dfrac{x+5}{7}$

202. $f(x) = \dfrac{2}{3}x + 2,\ g(x) = \dfrac{3}{2}x + 3$

203. $f(x) = \dfrac{1}{x-1},\ x \neq 1,\ g(x) = \dfrac{1}{x} + 1,\ x \neq 0$

204. $f(x) = x^3 + 1,\ g(x) = (x-1)^{1/3}$

205.
$f(x) = x^2 + 2x + 1,\ x \geq -1, \quad g(x) = -1 + \sqrt{x},\ x \geq 0$

206.
$f(x) = \sqrt{4 - x^2},\ 0 \leq x \leq 2,\ g(x) = \sqrt{4 - x^2},\ 0 \leq x \leq 2$

For the following exercises, evaluate the functions. Give the exact value.

207. $\tan^{-1}\left(\dfrac{\sqrt{3}}{3}\right)$

208. $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

209. $\cot^{-1}(1)$

210. $\sin^{-1}(-1)$

211. $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

212. $\cos\left(\tan^{-1}\left(\sqrt{3}\right)\right)$

213. $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

214. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

215. $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

216. The function $C = T(F) = (5/9)(F - 32)$ converts degrees Fahrenheit to degrees Celsius.

a. Find the inverse function $F = T^{-1}(C)$

b. What is the inverse function used for?

217. **[T]** The velocity V (in centimeters per second) of blood in an artery at a distance x cm from the center of the artery can be modeled by the function $V = f(x) = 500(0.04 - x^2)$ for $0 \le x \le 0.2$.

a. Find $x = f^{-1}(V)$.

b. Interpret what the inverse function is used for.

c. Find the distance from the center of an artery with a velocity of 15 cm/sec, 10 cm/sec, and 5 cm/sec.

218. A function that converts dress sizes in the United States to those in Europe is given by $D(x) = 2x + 24$.

a. Find the European dress sizes that correspond to sizes 6, 8, 10, and 12 in the United States.

b. Find the function that converts European dress sizes to U.S. dress sizes.

c. Use part b. to find the dress sizes in the United States that correspond to 46, 52, 62, and 70.

219. **[T]** The cost to remove a toxin from a lake is modeled by the function $C(p) = 75p/(85 - p)$, where C is the cost (in thousands of dollars) and p is the amount of toxin in a small lake (measured in parts per billion [ppb]). This model is valid only when the amount of toxin is less than 85 ppb.

a. Find the cost to remove 25 ppb, 40 ppb, and 50 ppb of the toxin from the lake.

b. Find the inverse function. c. Use part b. to determine how much of the toxin is removed for \$50,000.

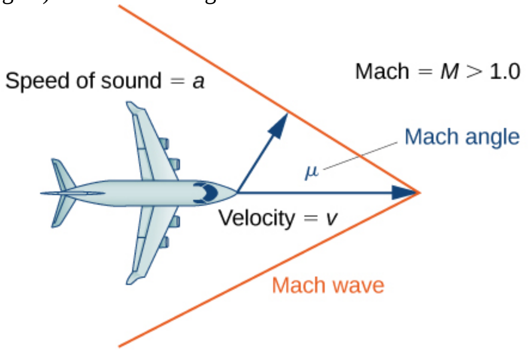
220. **[T]** A race car is accelerating at a velocity given by $v(t) = \frac{25}{4}t + 54$, where v is the velocity (in feet per second) at time t.

a. Find the velocity of the car at 10 sec.

b. Find the inverse function.

c. Use part b. to determine how long it takes for the car to reach a speed of 150 ft/sec.

221. **[T]** An airplane's Mach number M is the ratio of its speed to the speed of sound. When a plane is flying at a constant altitude, then its Mach angle is given by $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$. Find the Mach angle (to the nearest degree) for the following Mach numbers.



a. $M = 1.4$

b. $M = 2.8$

c. $M = 4.3$

222. **[T]** Using $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$, find the Mach number M for the following angles.

a. $\mu = \frac{\pi}{6}$

b. $\mu = \frac{2\pi}{7}$

c. $\mu = \frac{3\pi}{8}$

223. **[T]** The average temperature (in degrees Celsius) of a city in the northern United States can be modeled by the function $T(x) = 5 + 18\sin\left[\frac{\pi}{6}(x - 4.6)\right]$, where x is time in months and $x = 1.00$ corresponds to January 1. Determine the month and day when the average temperature is $21°\text{C}$.

224. **[T]** The depth (in feet) of water at a dock changes with the rise and fall of tides. It is modeled by the function $D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Determine the first time after midnight when the depth is 11.75 ft.

225. **[T]** An object moving in simple harmonic motion is modeled by the function $s(t) = -6\cos\left(\frac{\pi t}{2}\right)$, where s is measured in inches and t is measured in seconds. Determine the first time when the distance moved is 4.5 in.

226. **[T]** A local art gallery has a portrait 3 ft in height that is hung 2.5 ft above the eye level of an average person. The viewing angle θ can be modeled by the function $\theta = \tan^{-1}\frac{5.5}{x} - \tan^{-1}\frac{2.5}{x}$, where x is the distance (in feet) from the portrait. Find the viewing angle when a person is 4 ft from the portrait.

227. **[T]** Use a calculator to evaluate $\tan^{-1}(\tan(2.1))$ and $\cos^{-1}(\cos(2.1))$. Explain the results of each.

228. **[T]** Use a calculator to evaluate $\sin(\sin^{-1}(-2))$ and $\tan(\tan^{-1}(-2))$. Explain the results of each.

1.5 | Exponential and Logarithmic Functions

Learning Objectives

1.5.1 Identify the form of an exponential function.
1.5.2 Explain the difference between the graphs of x^b and b^x.
1.5.3 Recognize the significance of the number e.
1.5.4 Identify the form of a logarithmic function.
1.5.5 Explain the relationship between exponential and logarithmic functions.
1.5.6 Describe how to calculate a logarithm to a different base.
1.5.7 Identify the hyperbolic functions, their graphs, and basic identities.

In this section we examine exponential and logarithmic functions. We use the properties of these functions to solve equations involving exponential or logarithmic terms, and we study the meaning and importance of the number e. We also define hyperbolic and inverse hyperbolic functions, which involve combinations of exponential and logarithmic functions. (Note that we present alternative definitions of exponential and logarithmic functions in the chapter **Applications of Integrations**, and prove that the functions have the same properties with either definition.)

Exponential Functions

Exponential functions arise in many applications. One common example is population growth.

For example, if a population starts with P_0 individuals and then grows at an annual rate of 2%, its population after 1 year is

$$P(1) = P_0 + 0.02P_0 = P_0(1 + 0.02) = P_0(1.02).$$

Its population after 2 years is

$$P(2) = P(1) + 0.02P(1) = P(1)(1.02) = P_0(1.02)^2.$$

In general, its population after t years is

$$P(t) = P_0(1.02)^t,$$

which is an exponential function. More generally, any function of the form $f(x) = b^x$, where $b > 0, b \neq 1$, is an exponential function with **base** b and **exponent** x. Exponential functions have constant bases and variable exponents. Note that a function of the form $f(x) = x^b$ for some constant b is not an exponential function but a power function.

To see the difference between an exponential function and a power function, we compare the functions $y = x^2$ and $y = 2^x$. In **Table 1.10**, we see that both 2^x and x^2 approach infinity as $x \to \infty$. Eventually, however, 2^x becomes larger than x^2 and grows more rapidly as $x \to \infty$. In the opposite direction, as $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$. The line $y = 0$ is a horizontal asymptote for $y = 2^x$.

x	-3	-2	-1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
2^x	1/8	1/4	1/2	1	2	4	8	16	32	64

Table 1.10 Values of x^2 and 2^x

In **Figure 1.43**, we graph both $y = x^2$ and $y = 2^x$ to show how the graphs differ.

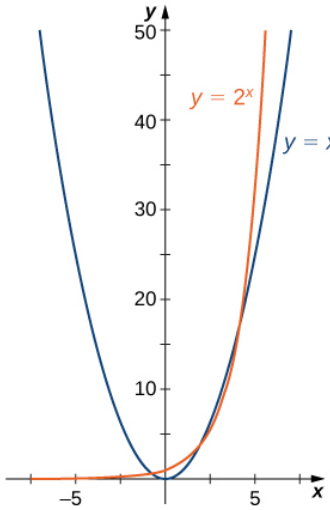


Figure 1.43 Both 2^x and x^2 approach infinity as $x \to \infty$, but 2^x grows more rapidly than x^2. As $x \to -\infty$, $x^2 \to \infty$, whereas $2^x \to 0$.

Evaluating Exponential Functions

Recall the properties of exponents: If x is a positive integer, then we define $b^x = b \cdot b \cdots b$ (with x factors of b). If x is a negative integer, then $x = -y$ for some positive integer y, and we define $b^x = b^{-y} = 1/b^y$. Also, b^0 is defined to be 1. If x is a rational number, then $x = p/q$, where p and q are integers and $b^x = b^{p/q} = \sqrt[q]{b^p}$. For example, $9^{3/2} = \sqrt{9^3} = 27$. However, how is b^x defined if x is an irrational number? For example, what do we mean by $2^{\sqrt{2}}$? This is too complex a question for us to answer fully right now; however, we can make an approximation. In **Table 1.11**, we list some rational numbers approaching $\sqrt{2}$, and the values of 2^x for each rational number x are presented as well. We claim that if we choose rational numbers x getting closer and closer to $\sqrt{2}$, the values of 2^x get closer and closer to some number L. We define that number L to be $2^{\sqrt{2}}$.

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
2^x	2.639	2.65737	2.66475	2.665119	2.665138	2.665143

Table 1.11 Values of 2^x for a List of Rational Numbers Approximating $\sqrt{2}$

Example 1.33

Bacterial Growth

Suppose a particular population of bacteria is known to double in size every 4 hours. If a culture starts with 1000 bacteria, the number of bacteria after 4 hours is $n(4) = 1000 \cdot 2$. The number of bacteria after 8 hours is $n(8) = n(4) \cdot 2 = 1000 \cdot 2^2$. In general, the number of bacteria after $4m$ hours is $n(4m) = 1000 \cdot 2^m$. Letting

$t = 4m$, we see that the number of bacteria after t hours is $n(t) = 1000 \cdot 2^{t/4}$. Find the number of bacteria after 6 hours, 10 hours, and 24 hours.

Solution

The number of bacteria after 6 hours is given by $n(6) = 1000 \cdot 2^{6/4} \approx 2828$ bacteria. The number of bacteria after 10 hours is given by $n(10) = 1000 \cdot 2^{10/4} \approx 5657$ bacteria. The number of bacteria after 24 hours is given by $n(24) = 1000 \cdot 2^{6} = 64{,}000$ bacteria.

1.27 Given the exponential function $f(x) = 100 \cdot 3^{x/2}$, evaluate $f(4)$ and $f(10)$.

Go to **World Population Balance (http://www.openstax.org/l/20_exponengrow)** for another example of exponential population growth.

Graphing Exponential Functions

For any base $b > 0, b \neq 1$, the exponential function $f(x) = b^x$ is defined for all real numbers x and $b^x > 0$. Therefore, the domain of $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. To graph b^x, we note that for $b > 1$, b^x is increasing on $(-\infty, \infty)$ and $b^x \to \infty$ as $x \to \infty$, whereas $b^x \to 0$ as $x \to -\infty$. On the other hand, if $0 < b < 1$, $f(x) = b^x$ is decreasing on $(-\infty, \infty)$ and $b^x \to 0$ as $x \to \infty$ whereas $b^x \to \infty$ as $x \to -\infty$ (**Figure 1.44**).

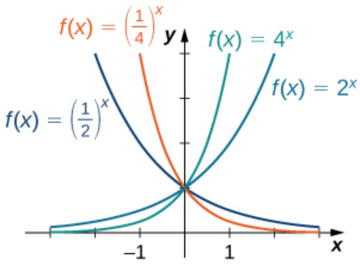


Figure 1.44 If $b > 1$, then b^x is increasing on $(-\infty, \infty)$. If $0 < b < 1$, then b^x is decreasing on $(-\infty, \infty)$.

Visit this **site (http://www.openstax.org/l/20_inverse)** for more exploration of the graphs of exponential functions.

Note that exponential functions satisfy the general laws of exponents. To remind you of these laws, we state them as rules.

Rule: Laws of Exponents

For any constants $a > 0$, $b > 0$, and for all x and y,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$

3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

Example 1.34

Using the Laws of Exponents

Use the laws of exponents to simplify each of the following expressions.

a. $\dfrac{\left(2x^{2/3}\right)^3}{\left(4x^{-1/3}\right)^2}$

b. $\dfrac{\left(x^3 y^{-1}\right)^2}{\left(xy^2\right)^{-2}}$

Solution

a. We can simplify as follows:

$$\frac{\left(2x^{2/3}\right)^3}{\left(4x^{-1/3}\right)^2} = \frac{2^3\left(x^{2/3}\right)^3}{4^2\left(x^{-1/3}\right)^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2 x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

b. We can simplify as follows:

$$\frac{\left(x^3 y^{-1}\right)^2}{\left(xy^2\right)^{-2}} = \frac{\left(x^3\right)^2\left(y^{-1}\right)^2}{x^{-2}\left(y^2\right)^{-2}} = \frac{x^6 y^{-2}}{x^{-2} y^{-4}} = x^6 x^2 y^{-2} y^4 = x^8 y^2.$$

1.28 Use the laws of exponents to simplify $\left(6x^{-3} y^2\right)/\left(12x^{-4} y^5\right)$.

The Number *e*

A special type of exponential function appears frequently in real-world applications. To describe it, consider the following example of exponential growth, which arises from compounding interest in a savings account. Suppose a person invests P dollars in a savings account with an annual interest rate r, compounded annually. The amount of money after 1 year is

$$A(1) = P + rP = P(1 + r).$$

The amount of money after 2 years is

$$A(2) = A(1) + rA(1) = P(1 + r) + rP(1 + r) = P(1 + r)^2.$$

More generally, the amount after t years is

$$A(t) = P(1 + r)^t.$$

If the money is compounded 2 times per year, the amount of money after half a year is

$$A\left(\frac{1}{2}\right) = P + \left(\frac{r}{2}\right)P = P\left(1 + \left(\frac{r}{2}\right)\right).$$

The amount of money after 1 year is

$$A(1) = A\left(\frac{1}{2}\right) + \left(\frac{r}{2}\right)A\left(\frac{1}{2}\right) = P\left(1 + \frac{r}{2}\right) + \frac{r}{2}\left(P\left(1 + \frac{r}{2}\right)\right) = P\left(1 + \frac{r}{2}\right)^2.$$

After t years, the amount of money in the account is

$$A(t) = P\left(1 + \frac{r}{2}\right)^{2t}.$$

More generally, if the money is compounded n times per year, the amount of money in the account after t years is given by the function

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

What happens as $n \to \infty$? To answer this question, we let $m = n/r$ and write

$$\left(1 + \frac{r}{n}\right)^{nt} = \left(1 + \frac{1}{m}\right)^{mrt},$$

and examine the behavior of $(1 + 1/m)^m$ as $m \to \infty$, using a table of values (**Table 1.12**).

m	10	100	1000	10,000	100,000	1,000,000
$\left(1 + \frac{1}{m}\right)^m$	2.5937	2.7048	2.71692	2.71815	2.718268	2.718280

Table 1.12 Values of $\left(1 + \frac{1}{m}\right)^m$ as $m \to \infty$

Looking at this table, it appears that $(1 + 1/m)^m$ is approaching a number between 2.7 and 2.8 as $m \to \infty$. In fact, $(1 + 1/m)^m$ does approach some number as $m \to \infty$. We call this **number** $\boldsymbol{e}$. To six decimal places of accuracy,

$$e \approx 2.718282.$$

The letter e was first used to represent this number by the Swiss mathematician Leonhard Euler during the 1720s. Although Euler did not discover the number, he showed many important connections between e and logarithmic functions. We still use the notation e today to honor Euler's work because it appears in many areas of mathematics and because we can use it in many practical applications.

Returning to our savings account example, we can conclude that if a person puts P dollars in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. This function may be familiar. Since functions involving base e arise often in applications, we call the function $f(x) = e^x$ the **natural exponential function**. Not only is this function interesting because of the definition of the number e, but also, as discussed next, its graph has an important property.

Since $e > 1$, we know e^x is increasing on $(-\infty, \infty)$. In **Figure 1.45**, we show a graph of $f(x) = e^x$ along with a *tangent line* to the graph of at $x = 0$. We give a precise definition of tangent line in the next chapter; but, informally, we say a tangent line to a graph of f at $x = a$ is a line that passes through the point $(a, f(a))$ and has the same "slope" as f at that point . The function $f(x) = e^x$ is the only exponential function b^x with tangent line at $x = 0$ that has a slope of 1. As we see later in the text, having this property makes the natural exponential function the most simple exponential function to use in many instances.

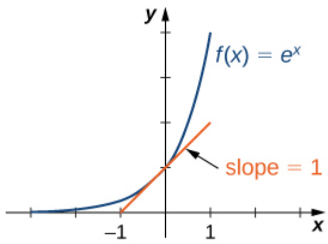


Figure 1.45 The graph of $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$.

Example 1.35

Compounding Interest

Suppose $\$500$ is invested in an account at an annual interest rate of $r = 5.5\%$, compounded continuously.

a. Let t denote the number of years after the initial investment and $A(t)$ denote the amount of money in the account at time t. Find a formula for $A(t)$.

b. Find the amount of money in the account after 10 years and after 20 years.

Solution

a. If P dollars are invested in an account at an annual interest rate r, compounded continuously, then $A(t) = Pe^{rt}$. Here $P = \$500$ and $r = 0.055$. Therefore, $A(t) = 500e^{0.055t}$.

b. After 10 years, the amount of money in the account is

$$A(10) = 500e^{0.055 \cdot 10} = 500e^{0.55} \approx \$866.63.$$

After 20 years, the amount of money in the account is

$$A(20) = 500e^{0.055 \cdot 20} = 500e^{1.1} \approx \$1,502.08.$$

1.29 If $\$750$ is invested in an account at an annual interest rate of 4%, compounded continuously, find a formula for the amount of money in the account after t years. Find the amount of money after 30 years.

Logarithmic Functions

Using our understanding of exponential functions, we can discuss their inverses, which are the logarithmic functions. These come in handy when we need to consider any phenomenon that varies over a wide range of values, such as pH in chemistry or decibels in sound levels.

The exponential function $f(x) = b^x$ is one-to-one, with domain $(-\infty, \infty)$ and range $(0, \infty)$. Therefore, it has an inverse function, called the *logarithmic function with base* b. For any $b > 0, b \neq 1$, the logarithmic function with base b, denoted $\log_b$, has domain $(0, \infty)$ and range $(-\infty, \infty)$, and satisfies

$$\log_b(x) = y \text{ if and only if } b^y = x.$$

For example,

$$\begin{array}{lll} \log_2(8) = 3 & \text{since} & 2^3 = 8, \\ \log_{10}\left(\frac{1}{100}\right) = -2 & \text{since} & 10^{-2} = \frac{1}{10^2} = \frac{1}{100}, \\ \log_b(1) = 0 & \text{since} & b^0 = 1 \text{ for any base } b > 0. \end{array}$$

Furthermore, since $y = \log_b(x)$ and $y = b^x$ are inverse functions,

$$\log_b(b^x) = x \text{ and } b^{\log_b(x)} = x.$$

The most commonly used logarithmic function is the function $\log_e$. Since this function uses natural e as its base, it is called the **natural logarithm**. Here we use the notation $\ln(x)$ or $\ln x$ to mean $\log_e(x)$. For example,

$$\ln(e) = \log_e(e) = 1,\ \ln\left(e^3\right) = \log_e\left(e^3\right) = 3,\ \ln(1) = \log_e(1) = 0.$$

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

and their graphs are symmetric about the line $y = x$ (**Figure 1.46**).

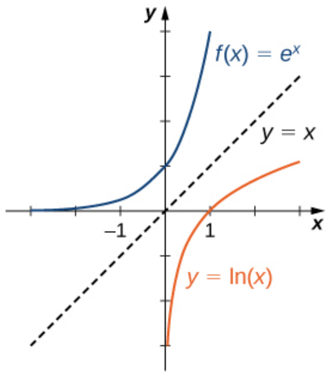


Figure 1.46 The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$.

At this **site (http://www.openstax.org/l/20_logscale)** you can see an example of a base-10 logarithmic scale.

In general, for any base $b > 0,\ b \neq 1,$ the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of the exponential functions, we graph functions $\log_b$ for several values of $b > 1$ (**Figure 1.47**).

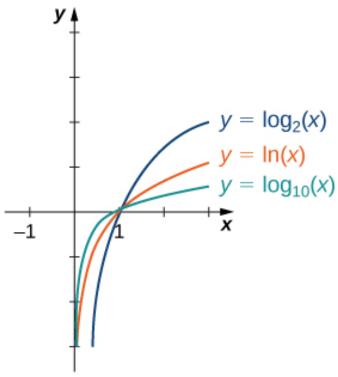


Figure 1.47 Graphs of $y = \log_b(x)$ are depicted for $b = 2,\ e,\ 10.$

Before solving some equations involving exponential and logarithmic functions, let's review the basic properties of logarithms.

Rule: Properties of Logarithms

If $a, b, c > 0, b \neq 1,$ and r is any real number, then

1. $\log_b(ac) = \log_b(a) + \log_b(c)$ (Product property)
2. $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$ (Quotient property)
3. $\log_b(a^r) = r\log_b(a)$ (Power property)

Example 1.36

Solving Equations Involving Exponential Functions

Solve each of the following equations for x.

a. $5^x = 2$

b. $e^x + 6e^{-x} = 5$

Solution

a. Applying the natural logarithm function to both sides of the equation, we have

$$\ln 5^x = \ln 2.$$

Using the power property of logarithms,

$$x \ln 5 = \ln 2.$$

Therefore, $x = \ln 2/\ln 5$.

b. Multiplying both sides of the equation by e^x, we arrive at the equation

$$e^{2x} + 6 = 5e^x.$$

Rewriting this equation as

$$e^{2x} - 5e^x + 6 = 0,$$

we can then rewrite it as a quadratic equation in e^x:

$$(e^x)^2 - 5(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

$$(e^x - 3)(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

1.30 Solve $e^{2x}/(3 + e^{2x}) = 1/2$.

Example 1.37

Solving Equations Involving Logarithmic Functions

Solve each of the following equations for x.

a. $\ln\left(\frac{1}{x}\right) = 4$

b. $\log_{10}\sqrt{x} + \log_{10}x = 2$

c. $\ln(2x) - 3\ln\left(x^2\right) = 0$

Solution

a. By the definition of the natural logarithm function,

$$\ln\left(\frac{1}{x}\right) = 4 \text{ if and only if } e^4 = \frac{1}{x}.$$

Therefore, the solution is $x = 1/e^4$.

b. Using the product and power properties of logarithmic functions, rewrite the left-hand side of the equation as

$$\log_{10}\sqrt{x} + \log_{10}x = \log_{10}x\sqrt{x} = \log_{10}x^{3/2} = \frac{3}{2}\log_{10}x.$$

Therefore, the equation can be rewritten as

$$\frac{3}{2}\log_{10}x = 2 \text{ or } \log_{10}x = \frac{4}{3}.$$

The solution is $x = 10^{4/3} = 10\sqrt[3]{10}.$

c. Using the power property of logarithmic functions, we can rewrite the equation as $\ln(2x) - \ln\left(x^6\right) = 0.$ Using the quotient property, this becomes

$$\ln\left(\frac{2}{x^5}\right) = 0.$$

Therefore, $2/x^5 = 1,$ which implies $x = \sqrt[5]{2}.$ We should then check for any extraneous solutions.

1.31 Solve $\ln\left(x^3\right) - 4\ln(x) = 1.$

When evaluating a logarithmic function with a calculator, you may have noticed that the only options are $\log_{10}$ or log, called the *common logarithm*, or *ln*, which is the natural logarithm. However, exponential functions and logarithm functions can be expressed in terms of any desired base $b.$ If you need to use a calculator to evaluate an expression with a different base, you can apply the change-of-base formulas first. Using this change of base, we typically write a given exponential or logarithmic function in terms of the natural exponential and natural logarithmic functions.

Rule: Change-of-Base Formulas

Let $a > 0,\ b > 0,$ and $a \neq 1,\ b \neq 1.$

1. $a^x = b^{x\log_b a}$ for any real number $x.$

   If $b = e,$ this equation reduces to $a^x = e^{x\log_e a} = e^{x\ln a}.$

2. $\log_a x = \dfrac{\log_b x}{\log_b a}$ for any real number $x > 0.$

   If $b = e,$ this equation reduces to $\log_a x = \dfrac{\ln x}{\ln a}.$

Proof

For the first change-of-base formula, we begin by making use of the power property of logarithmic functions. We know that for any base $b > 0,\ b \neq 1,\ \log_b(a^x) = x\log_b a.$ Therefore,

$$b^{\log_b(a^x)} = b^{x\log_b a}.$$

In addition, we know that b^x and $\log_b(x)$ are inverse functions. Therefore,

$$b^{\log_b(a^x)} = a^x.$$

Combining these last two equalities, we conclude that $a^x = b^{x\log_b a}.$

To prove the second property, we show that

$$(\log_b a) \cdot (\log_a x) = \log_b x.$$

Let $u = \log_b a,\ v = \log_a x,$ and $w = \log_b x.$ We will show that $u \cdot v = w.$ By the definition of logarithmic functions, we know that $b^u = a$, $a^v = x$, and $b^w = x$. From the previous equations, we see that

$$b^{uv} = (b^u)^v = a^v = x = b^w.$$

Therefore, $b^{uv} = b^w$. Since exponential functions are one-to-one, we can conclude that $u \cdot v = w$.

□

Example 1.38

Changing Bases

Use a calculating utility to evaluate $\log_3 7$ with the change-of-base formula presented earlier.

Solution

Use the second equation with $a = 3$ and $e = 3$:

$$\log_3 7 = \frac{\ln 7}{\ln 3} \approx 1.77124.$$

1.32 Use the change-of-base formula and a calculating utility to evaluate $\log_4 6$.

Example 1.39

Chapter Opener: The Richter Scale for Earthquakes

Figure 1.48 (credit: modification of work by Robb Hannawacker, NPS)

In 1935, Charles Richter developed a scale (now known as the *Richter scale*) to measure the magnitude of an earthquake. The scale is a base-10 logarithmic scale, and it can be described as follows: Consider one earthquake with magnitude R_1 on the Richter scale and a second earthquake with magnitude R_2 on the Richter scale. Suppose $R_1 > R_2$, which means the earthquake of magnitude R_1 is stronger, but how much stronger is it than the other earthquake? A way of measuring the intensity of an earthquake is by using a seismograph to measure the amplitude of the earthquake waves. If A_1 is the amplitude measured for the first earthquake and A_2 is the amplitude measured for the second earthquake, then the amplitudes and magnitudes of the two earthquakes satisfy the following equation:

$$R_1 - R_2 = \log_{10}\left(\frac{A_1}{A_2}\right).$$

Consider an earthquake that measures 8 on the Richter scale and an earthquake that measures 7 on the Richter scale. Then,

$$8 - 7 = \log_{10}\left(\frac{A_1}{A_2}\right).$$

Therefore,

$$\log_{10}\left(\frac{A_1}{A_2}\right) = 1,$$

which implies $A_1/A_2 = 10$ or $A_1 = 10A_2$. Since A_1 is 10 times the size of A_2, we say that the first earthquake is 10 times as intense as the second earthquake. On the other hand, if one earthquake measures 8 on the Richter scale and another measures 6, then the relative intensity of the two earthquakes satisfies the equation

$$\log_{10}\left(\frac{A_1}{A_2}\right) = 8 - 6 = 2.$$

Therefore, $A_1 = 100A_2$. That is, the first earthquake is 100 times more intense than the second earthquake.

How can we use logarithmic functions to compare the relative severity of the magnitude 9 earthquake in Japan in 2011 with the magnitude 7.3 earthquake in Haiti in 2010?

Solution

To compare the Japan and Haiti earthquakes, we can use an equation presented earlier:

$$9 - 7.3 = \log_{10}\left(\frac{A_1}{A_2}\right).$$

Therefore, $A_1/A_2 = 10^{1.7}$, and we conclude that the earthquake in Japan was approximately 50 times more intense than the earthquake in Haiti.

1.33 Compare the relative severity of a magnitude 8.4 earthquake with a magnitude 7.4 earthquake.

Hyperbolic Functions

The hyperbolic functions are defined in terms of certain combinations of e^x and e^{-x}. These functions arise naturally in various engineering and physics applications, including the study of water waves and vibrations of elastic membranes. Another common use for a hyperbolic function is the representation of a hanging chain or cable, also known as a catenary (**Figure 1.49**). If we introduce a coordinate system so that the low point of the chain lies along the y-axis, we can describe the height of the chain in terms of a hyperbolic function. First, we define the **hyperbolic functions**.

Figure 1.49 The shape of a strand of silk in a spider's web can be described in terms of a hyperbolic function. The same shape applies to a chain or cable hanging from two supports with only its own weight. (credit: "Mtpaley", Wikimedia Commons)

Definition

Hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic secant

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cotangent

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The name *cosh* rhymes with "gosh," whereas the name *sinh* is pronounced "cinch." *Tanh*, *sech*, *csch*, and *coth* are pronounced "tanch," "seech," "coseech," and "cotanch," respectively.

Using the definition of $\cosh(x)$ and principles of physics, it can be shown that the height of a hanging chain, such as the one in **Figure 1.49**, can be described by the function $h(x) = a\cosh(x/a) + c$ for certain constants a and c.

But why are these functions called *hyperbolic functions*? To answer this question, consider the quantity $\cosh^2 t - \sinh^2 t$. Using the definition of $\cosh$ and $\sinh$, we see that

$$\cosh^2 t - \sinh^2 t = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = 1.$$

This identity is the analog of the trigonometric identity $\cos^2 t + \sin^2 t = 1$. Here, given a value t, the point $(x, y) = (\cosh t, \sinh t)$ lies on the unit hyperbola $x^2 - y^2 = 1$ (**Figure 1.50**).

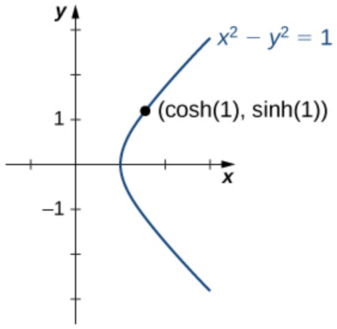


Figure 1.50 The unit hyperbola $\cosh^2 t - \sinh^2 t = 1$.

Graphs of Hyperbolic Functions

To graph $\cosh x$ and $\sinh x$, we make use of the fact that both functions approach $(1/2)e^x$ as $x \to \infty$, since $e^{-x} \to 0$ as $x \to \infty$. As $x \to -\infty$, $\cosh x$ approaches $1/2e^{-x}$, whereas $\sinh x$ approaches $-1/2e^{-x}$. Therefore, using the graphs of $1/2e^x$, $1/2e^{-x}$, and $-1/2e^{-x}$ as guides, we graph $\cosh x$ and $\sinh x$. To graph $\tanh x$, we use the fact that $\tanh(0) = 0$, $-1 < \tanh(x) < 1$ for all x, $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$ as $x \to -\infty$. The graphs of the other three hyperbolic functions can be sketched using the graphs of $\cosh x$, $\sinh x$, and $\tanh x$ (**Figure 1.51**).

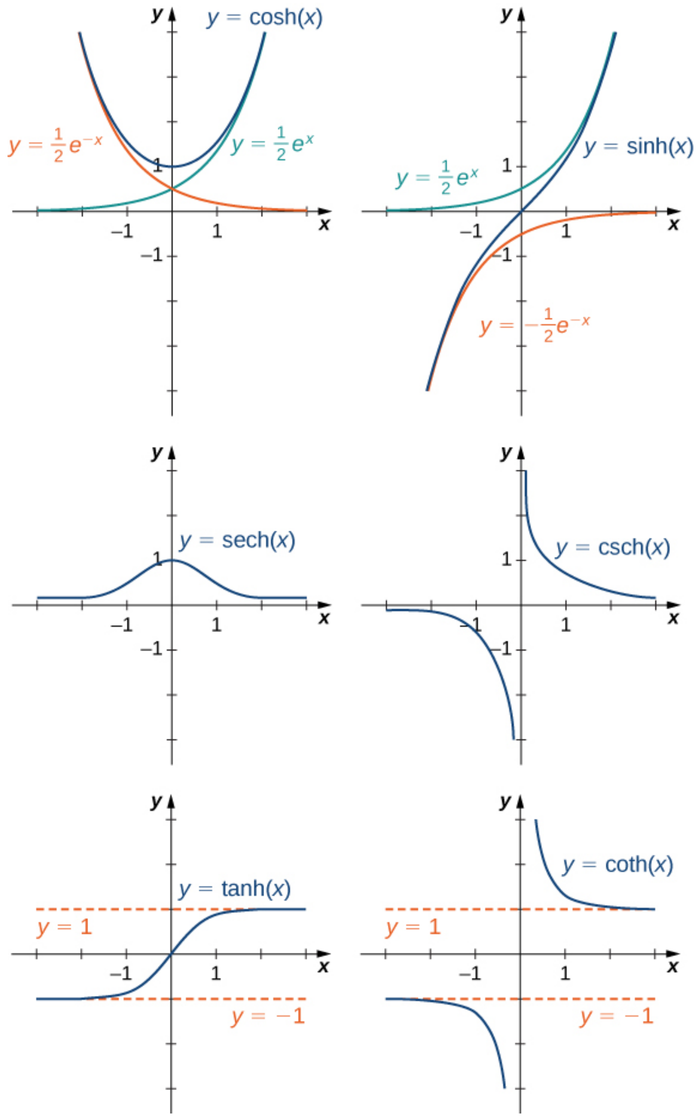


Figure 1.51 The hyperbolic functions involve combinations of e^x and e^{-x}.

Identities Involving Hyperbolic Functions

The identity $\cosh^2 t - \sinh^2 t$, shown in **Figure 1.50**, is one of several identities involving the hyperbolic functions, some of which are listed next. The first four properties follow easily from the definitions of hyperbolic sine and hyperbolic cosine. Except for some differences in signs, most of these properties are analogous to identities for trigonometric functions.

Rule: Identities Involving Hyperbolic Functions

1. $\cosh(-x) = \cosh x$
2. $\sinh(-x) = -\sinh x$
3. $\cosh x + \sinh x = e^x$
4. $\cosh x - \sinh x = e^{-x}$

5. $\cosh^2 x - \sinh^2 x = 1$
6. $1 - \tanh^2 x = \operatorname{sech}^2 x$
7. $\coth^2 x - 1 = \operatorname{csch}^2 x$
8. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
9. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

Example 1.40

Evaluating Hyperbolic Functions

a. Simplify $\sinh(5 \ln x)$.

b. If $\sinh x = 3/4$, find the values of the remaining five hyperbolic functions.

Solution

a. Using the definition of the $\sinh$ function, we write

$$\sinh(5 \ln x) = \frac{e^{5 \ln x} - e^{-5 \ln x}}{2} = \frac{e^{\ln\left(x^5\right)} - e^{\ln\left(x^{-5}\right)}}{2} = \frac{x^5 - x^{-5}}{2}.$$

b. Using the identity $\cosh^2 x - \sinh^2 x = 1$, we see that

$$\cosh^2 x = 1 + \left(\frac{3}{4}\right)^2 = \frac{25}{16}.$$

Since $\cosh x \geq 1$ for all x, we must have $\cosh x = 5/4$. Then, using the definitions for the other hyperbolic functions, we conclude that $\tanh x = 3/5$, $\operatorname{csch} x = 4/3$, $\operatorname{sech} x = 4/5$, and $\coth x = 5/3$.

1.34 Simplify $\cosh(2 \ln x)$.

Inverse Hyperbolic Functions

From the graphs of the hyperbolic functions, we see that all of them are one-to-one except $\cosh x$ and $\operatorname{sech} x$. If we restrict the domains of these two functions to the interval $[0, \infty)$, then all the hyperbolic functions are one-to-one, and we can define the **inverse hyperbolic functions**. Since the hyperbolic functions themselves involve exponential functions, the inverse hyperbolic functions involve logarithmic functions.

Definition

Inverse Hyperbolic Functions

$$\sinh^{-1} x = \operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad \cosh^{-1} x = \operatorname{arccosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\tanh^{-1} x = \operatorname{arctanh} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \qquad \coth^{-1} x = \operatorname{arccot} x = \frac{1}{2}\ln\left(\frac{x + 1}{x - 1}\right)$$

$$\operatorname{sech}^{-1} x = \operatorname{arcsech} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) \qquad \operatorname{csch}^{-1} x = \operatorname{arccsch} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|}\right)$$

Let's look at how to derive the first equation. The others follow similarly. Suppose $y = \sinh^{-1} x.$ Then, $x = \sinh y$ and, by the definition of the hyperbolic sine function, $x = \frac{e^y - e^{-y}}{2}.$ Therefore,

$$e^y - 2x - e^{-y} = 0.$$

Multiplying this equation by $e^y,$ we obtain

$$e^{2y} - 2xe^y - 1 = 0.$$

This can be solved like a quadratic equation, with the solution

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since $e^y > 0,$ the only solution is the one with the positive sign. Applying the natural logarithm to both sides of the equation, we conclude that

$$y = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Example 1.41

Evaluating Inverse Hyperbolic Functions

Evaluate each of the following expressions.

$$\sinh^{-1}(2)$$
$$\tanh^{-1}(1/4)$$

Solution

$$\sinh^{-1}(2) = \ln\left(2 + \sqrt{2^2 + 1}\right) = \ln\left(2 + \sqrt{5}\right) \approx 1.4436$$

$$\tanh^{-1}(1/4) = \frac{1}{2}\ln\left(\frac{1 + 1/4}{1 - 1/4}\right) = \frac{1}{2}\ln\left(\frac{5/4}{3/4}\right) = \frac{1}{2}\ln\left(\frac{5}{3}\right) \approx 0.2554$$

1.35 Evaluate $\tanh^{-1}(1/2).$

1.5 EXERCISES

For the following exercises, evaluate the given exponential functions as indicated, accurate to two significant digits after the decimal.

229. $f(x) = 5^x$ a. $x = 3$ b. $x = \frac{1}{2}$ c. $x = \sqrt{2}$

230. $f(x) = (0.3)^x$ a. $x = -1$ b. $x = 4$ c. $x = -1.5$

231. $f(x) = 10^x$ a. $x = -2$ b. $x = 4$ c. $x = \frac{5}{3}$

232. $f(x) = e^x$ a. $x = 2$ b. $x = -3.2$ c. $x = \pi$

For the following exercises, match the exponential equation to the correct graph.

a. $y = 4^{-x}$

b. $y = 3^{x-1}$

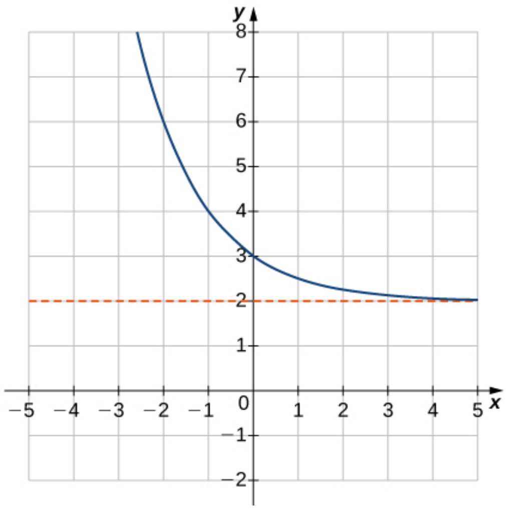
c. $y = 2^{x+1}$

d. $y = \left(\frac{1}{2}\right)^x + 2$

e. $y = -3^{-x}$

f. $y = 1 - 5^x$

233.



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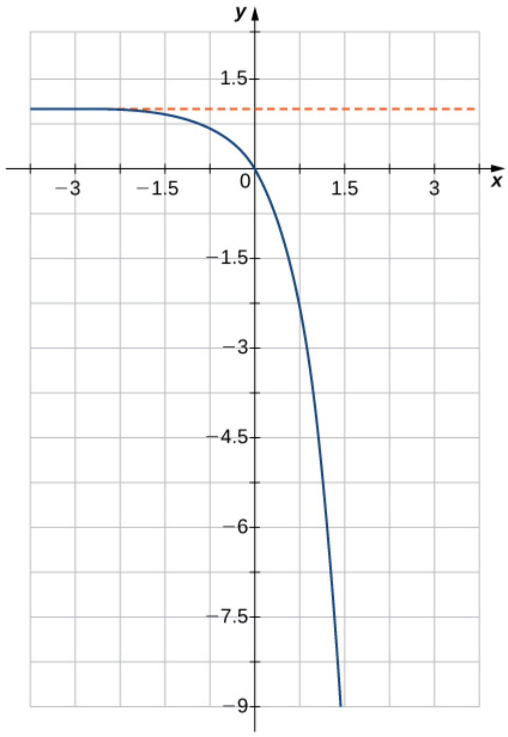

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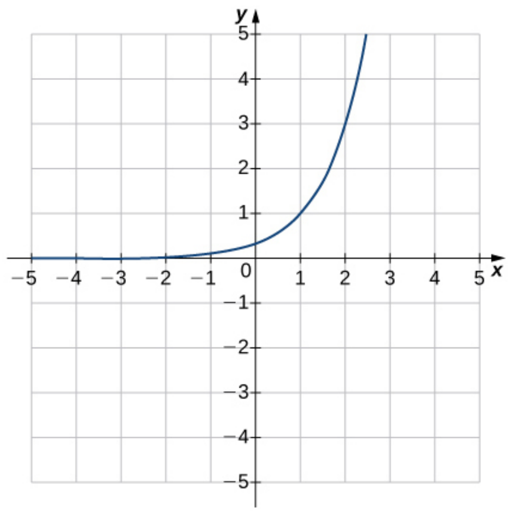

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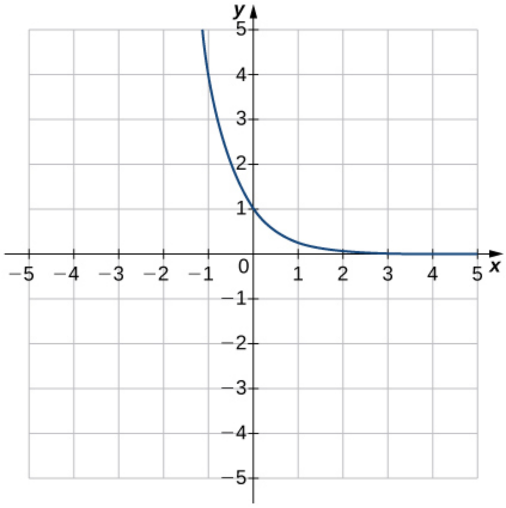

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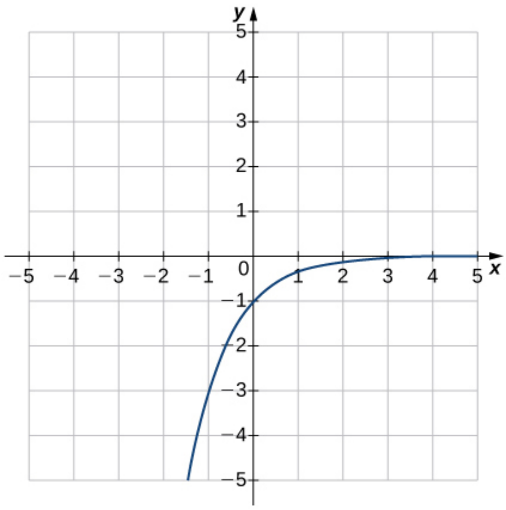

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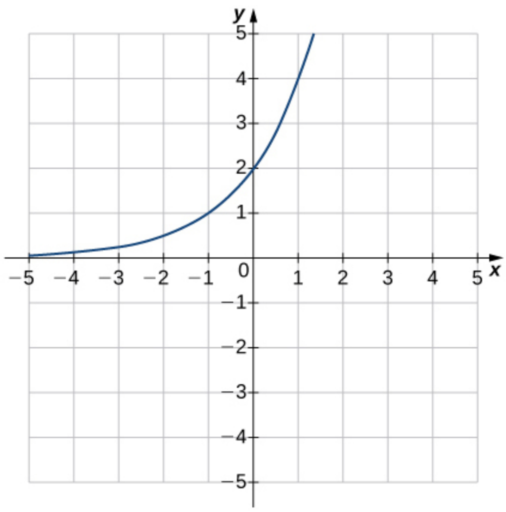

For the following exercises, sketch the graph of the exponential function. Determine the domain, range, and horizontal asymptote.

239. $f(x) = e^x + 2$

240. $f(x) = -2^x$

241. $f(x) = 3^{x+1}$

242. $f(x) = 4^x - 1$

243. $f(x) = 1 - 2^{-x}$

244. $f(x) = 5^{x+1} + 2$

245. $f(x) = e^{-x} - 1$

For the following exercises, write the equation in equivalent exponential form.

246. $\log_3 81 = 4$

247. $\log_8 2 = \frac{1}{3}$

248. $\log_5 1 = 0$

249. $\log_5 25 = 2$

250. $\log 0.1 = -1$

251. $\ln\left(\frac{1}{e^3}\right) = -3$

252. $\log_9 3 = 0.5$

253. $\ln 1 = 0$

For the following exercises, write the equation in equivalent logarithmic form.

254. $2^3 = 8$

255. $4^{-2} = \frac{1}{16}$

256. $10^2 = 100$

257. $9^0 = 1$

258. $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

259. $\sqrt[3]{64} = 4$

260. $e^x = y$

261. $9^y = 150$

262. $b^3 = 45$

263. $4^{-3/2} = 0.125$

For the following exercises, sketch the graph of the logarithmic function. Determine the domain, range, and vertical asymptote.

264. $f(x) = 3 + \ln x$

265. $f(x) = \ln(x - 1)$

266. $f(x) = \ln(-x)$

267. $f(x) = 1 - \ln x$

268. $f(x) = \log x - 1$

269. $f(x) = \ln(x + 1)$

For the following exercises, use properties of logarithms to write the expressions as a sum, difference, and/or product of logarithms.

270. $\log x^4 y$

271. $\log_3 \frac{9a^3}{b}$

272. $\ln a\sqrt[3]{b}$

273. $\log_5 \sqrt{125xy^3}$

274. $\log_4 \frac{\sqrt[3]{xy}}{64}$

275. $\ln\left(\frac{6}{\sqrt{e^3}}\right)$

For the following exercises, solve the exponential equation exactly.

276. $5^x = 125$

277. $e^{3x} - 15 = 0$

278. $8^x = 4$

279. $4^{x+1} - 32 = 0$

280. $3^{x/14} = \frac{1}{10}$

281. $10^x = 7.21$

282. $4 \cdot 2^{3x} - 20 = 0$

283. $7^{3x-2} = 11$

For the following exercises, solve the logarithmic equation exactly, if possible.

284. $\log_3 x = 0$

285. $\log_5 x = -2$

286. $\log_4 (x + 5) = 0$

287. $\log(2x - 7) = 0$

288. $\ln\sqrt{x + 3} = 2$

289. $\log_6 (x + 9) + \log_6 x = 2$

290. $\log_4 (x + 2) - \log_4 (x - 1) = 0$

291. $\ln x + \ln(x - 2) = \ln 4$

For the following exercises, use the change-of-base formula and either base 10 or base e to evaluate the given expressions. Answer in exact form and in approximate form, rounding to four decimal places.

292. $\log_5 47$

293. $\log_7 82$

294. $\log_6 103$

295. $\log_{0.5} 211$

296. $\log_2 \pi$

297. $\log_{0.2} 0.452$

298. Rewrite the following expressions in terms of exponentials and simplify. a. $2\cosh(\ln x)$ b. $\cosh 4x + \sinh 4x$ c. $\cosh 2x - \sinh 2x$ d. $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

299. **[T]** The number of bacteria N in a culture after t days can be modeled by the function $N(t) = 1300 \cdot (2)^{t/4}$. Find the number of bacteria present after 15 days.

300. **[T]** The demand D (in millions of barrels) for oil in an oil-rich country is given by the function $D(p) = 150 \cdot (2.7)^{-0.25p}$, where p is the price (in dollars) of a barrel of oil. Find the amount of oil demanded (to the nearest million barrels) when the price is between \$15 and \$20.

301. **[T]** The amount A of a \$100,000 investment paying continuously and compounded for t years is given by $A(t) = 100{,}000 \cdot e^{0.055t}$. Find the amount A accumulated in 5 years.

302. **[T]** An investment is compounded monthly, quarterly, or yearly and is given by the function $A = P\left(1 + \frac{j}{n}\right)^{nt}$, where A is the value of the investment at time t, P is the initial principle that was invested, j is the annual interest rate, and n is the number of time the interest is compounded per year. Given a yearly interest rate of 3.5% and an initial principle of \$100,000, find the amount A accumulated in 5 years for interest that is compounded a. daily, b., monthly, c. quarterly, and d. yearly.

303. **[T]** The concentration of hydrogen ions in a substance is denoted by $[\mathrm{H}^+]$, measured in moles per liter. The pH of a substance is defined by the logarithmic function $\mathrm{pH} = -\log[\mathrm{H}^+]$. This function is used to measure the acidity of a substance. The pH of water is 7. A substance with a pH less than 7 is an acid, whereas one that has a pH of more than 7 is a base.

a. Find the pH of the following substances. Round answers to one digit.
b. Determine whether the substance is an acid or a base.
   i. Eggs: $[\mathrm{H}^+] = 1.6 \times 10^{-8}$ mol/L
   ii. Beer: $[\mathrm{H}^+] = 3.16 \times 10^{-3}$ mol/L
   iii. Tomato Juice: $[\mathrm{H}^+] = 7.94 \times 10^{-5}$ mol/L

304. **[T]** Iodine-131 is a radioactive substance that decays according to the function $Q(t) = Q_0 \cdot e^{-0.08664t}$, where Q_0 is the initial quantity of a sample of the substance and t is in days. Determine how long it takes (to the nearest day) for 95% of a quantity to decay.

305. **[T]** According to the World Bank, at the end of 2013 ($t = 0$) the U.S. population was 316 million and was increasing according to the following model: $P(t) = 316e^{0.0074t}$, where P is measured in millions of people and t is measured in years after 2013.

a. Based on this model, what will be the population of the United States in 2020?
b. Determine when the U.S. population will be twice what it is in 2013.

306. **[T]** The amount A accumulated after 1000 dollars is invested for t years at an interest rate of 4% is modeled by the function $A(t) = 1000(1.04)^t$.

a. Find the amount accumulated after 5 years and 10 years.
b. Determine how long it takes for the original investment to triple.

307. **[T]** A bacterial colony grown in a lab is known to double in number in 12 hours. Suppose, initially, there are 1000 bacteria present.

a. Use the exponential function $Q = Q_0 e^{kt}$ to determine the value k, which is the growth rate of the bacteria. Round to four decimal places.
b. Determine approximately how long it takes for 200,000 bacteria to grow.

308. **[T]** The rabbit population on a game reserve doubles every 6 months. Suppose there were 120 rabbits initially.

a. Use the exponential function $P = P_0 a^t$ to determine the growth rate constant a. Round to four decimal places.
b. Use the function in part a. to determine approximately how long it takes for the rabbit population to reach 3500.

309. **[T]** The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time, in Japan, an earthquake with magnitude 4.9 caused only minor damage. Approximately how much more energy was released by the San Francisco earthquake than by the Japanese earthquake?

CHAPTER 1 REVIEW

KEY TERMS

absolute value function $f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

algebraic function a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable x

base the number b in the exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$

composite function given two functions f and g, a new function, denoted $g \circ f$, such that $(g \circ f)(x) = g(f(x))$

cubic function a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

decreasing on the interval *I* a function decreasing on the interval I if, for all $x_1, x_2 \in I$, $f(x_1) \geq f(x_2)$ if $x_1 < x_2$

degree for a polynomial function, the value of the largest exponent of any term

dependent variable the output variable for a function

domain the set of inputs for a function

even function a function is even if $f(-x) = f(x)$ for all x in the domain of f

exponent the value x in the expression b^x

function a set of inputs, a set of outputs, and a rule for mapping each input to exactly one output

graph of a function the set of points (x, y) such that x is in the domain of f and $y = f(x)$

horizontal line test a function f is one-to-one if and only if every horizontal line intersects the graph of f, at most, once

hyperbolic functions the functions denoted $\sinh, \cosh, \tanh, \operatorname{csch}, \operatorname{sech},$ and $\coth,$ which involve certain combinations of e^x and e^{-x}

increasing on the interval *I* a function increasing on the interval I if for all $x_1, x_2 \in I$, $f(x_1) \leq f(x_2)$ if $x_1 < x_2$

independent variable the input variable for a function

inverse function for a function f, the inverse function f^{-1} satisfies $f^{-1}(y) = x$ if $f(x) = y$

inverse hyperbolic functions the inverses of the hyperbolic functions where $\cosh$ and sech are restricted to the domain $[0, \infty)$; each of these functions can be expressed in terms of a composition of the natural logarithm function and an algebraic function

inverse trigonometric functions the inverses of the trigonometric functions are defined on restricted domains where they are one-to-one functions

linear function a function that can be written in the form $f(x) = mx + b$

logarithmic function a function of the form $f(x) = \log_b(x)$ for some base $b > 0, b \neq 1$ such that $y = \log_b(x)$ if and only if $b^y = x$

mathematical model A method of simulating real-life situations with mathematical equations

natural exponential function the function $f(x) = e^x$

natural logarithm the function $\ln x = \log_e x$

number e as m gets larger, the quantity $(1 + (1/m)^m$ gets closer to some real number; we define that real number to be e; the value of e is approximately 2.718282

odd function a function is odd if $f(-x) = -f(x)$ for all x in the domain of f

one-to-one function a function f is one-to-one if $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

periodic function a function is periodic if it has a repeating pattern as the values of x move from left to right

piecewise-defined function a function that is defined differently on different parts of its domain

point-slope equation equation of a linear function indicating its slope and a point on the graph of the function

polynomial function a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

power function a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

quadratic function a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$ where $a \neq 0$

radians for a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s

range the set of outputs for a function

rational function a function of the form $f(x) = p(x)/q(x),$ where $p(x)$ and $q(x)$ are polynomials

restricted domain a subset of the domain of a function f

root function a function of the form $f(x) = x^{1/n}$ for any integer $n \geq 2$

slope the change in *y* for each unit change in *x*

slope-intercept form equation of a linear function indicating its slope and *y*-intercept

symmetry about the origin the graph of a function f is symmetric about the origin if $(-x, -y)$ is on the graph of f whenever (x, y) is on the graph

symmetry about the *y*-axis the graph of a function f is symmetric about the y-axis if $(-x, y)$ is on the graph of f whenever (x, y) is on the graph

table of values a table containing a list of inputs and their corresponding outputs

transcendental function a function that cannot be expressed by a combination of basic arithmetic operations

transformation of a function a shift, scaling, or reflection of a function

trigonometric functions functions of an angle defined as ratios of the lengths of the sides of a right triangle

trigonometric identity an equation involving trigonometric functions that is true for all angles θ for which the functions in the equation are defined

vertical line test given the graph of a function, every vertical line intersects the graph, at most, once

zeros of a function when a real number x is a zero of a function f, $f(x) = 0$

KEY EQUATIONS

- **Composition of two functions**
  $(g \circ f)(x) = g(f(x))$
- **Absolute value function**
  $f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

- **Point-slope equation of a line**
  $y - y_1 = m(x - x_1)$
- **Slope-intercept form of a line**
  $y = mx + b$
- **Standard form of a line**
  $ax + by = c$
- **Polynomial function**
  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
- **Generalized sine function**
  $f(x) = A\sin(B(x - \alpha)) + C$
- **Inverse functions**
  $f^{-1}(f(x)) = x$ for all x in D, and $f\left(f^{-1}(y)\right) = y$ for all y in R.

KEY CONCEPTS

1.1 Review of Functions

- A function is a mapping from a set of inputs to a set of outputs with exactly one output for each input.
- If no domain is stated for a function $y = f(x)$, the domain is considered to be the set of all real numbers x for which the function is defined.
- When sketching the graph of a function f, each vertical line may intersect the graph, at most, once.
- A function may have any number of zeros, but it has, at most, one y-intercept.
- To define the composition $g \circ f$, the range of f must be contained in the domain of g.
- Even functions are symmetric about the y-axis whereas odd functions are symmetric about the origin.

1.2 Basic Classes of Functions

- The power function $f(x) = x^n$ is an even function if n is even and $n \neq 0$, and it is an odd function if n is odd.
- The root function $f(x) = x^{1/n}$ has the domain $[0, \infty)$ if n is even and the domain $(-\infty, \infty)$ if n is odd. If n is odd, then $f(x) = x^{1/n}$ is an odd function.
- The domain of the rational function $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomial functions, is the set of x such that $q(x) \neq 0$.
- Functions that involve the basic operations of addition, subtraction, multiplication, division, and powers are algebraic functions. All other functions are transcendental. Trigonometric, exponential, and logarithmic functions are examples of transcendental functions.
- A polynomial function f with degree $n \geq 1$ satisfies $f(x) \to \pm\infty$ as $x \to \pm\infty$. The sign of the output as $x \to \infty$ depends on the sign of the leading coefficient only and on whether n is even or odd.
- Vertical and horizontal shifts, vertical and horizontal scalings, and reflections about the x- and y-axes are examples of transformations of functions.

1.3 Trigonometric Functions

- Radian measure is defined such that the angle associated with the arc of length 1 on the unit circle has radian measure 1. An angle with a degree measure of $180°$ has a radian measure of π rad.

- For acute angles θ, the values of the trigonometric functions are defined as ratios of two sides of a right triangle in which one of the acute angles is θ.
- For a general angle θ, let (x, y) be a point on a circle of radius r corresponding to this angle θ. The trigonometric functions can be written as ratios involving x, y, and r.
- The trigonometric functions are periodic. The sine, cosine, secant, and cosecant functions have period 2π. The tangent and cotangent functions have period π.

1.4 Inverse Functions

- For a function to have an inverse, the function must be one-to-one. Given the graph of a function, we can determine whether the function is one-to-one by using the horizontal line test.
- If a function is not one-to-one, we can restrict the domain to a smaller domain where the function is one-to-one and then define the inverse of the function on the smaller domain.
- For a function f and its inverse f^{-1}, $f\left(f^{-1}(x)\right) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f.
- Since the trigonometric functions are periodic, we need to restrict their domains to define the inverse trigonometric functions.
- The graph of a function f and its inverse f^{-1} are symmetric about the line $y = x$.

1.5 Exponential and Logarithmic Functions

- The exponential function $y = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$. Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- The logarithmic function $y = \log_b(x)$ is the inverse of $y = b^x$. Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- The natural exponential function is $y = e^x$ and the natural logarithmic function is $y = \ln x = \log_e x$.
- Given an exponential function or logarithmic function in base a, we can make a change of base to convert this function to any base $b > 0$, $b \neq 1$. We typically convert to base e.
- The hyperbolic functions involve combinations of the exponential functions e^x and e^{-x}. As a result, the inverse hyperbolic functions involve the natural logarithm.

CHAPTER 1 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample.

310. A function is always one-to-one.

311. $f \circ g = g \circ f$, assuming *f* and *g* are functions.

312. A relation that passes the horizontal and vertical line tests is a one-to-one function.

313. A relation passing the horizontal line test is a function.

For the following problems, state the domain and range of the given functions:

$f = x^2 + 2x - 3, \quad g = \ln(x - 5), \quad h = \dfrac{1}{x + 4}$

314. h

315. g

316. $h \circ f$

317. $g \circ f$

Find the degree, *y*-intercept, and zeros for the following polynomial functions.

318. $f(x) = 2x^2 + 9x - 5$

319. $f(x) = x^3 + 2x^2 - 2x$

Simplify the following trigonometric expressions.

320. $\frac{\tan^2 x}{\sec^2 x} + \cos^2 x$

321. $\cos^2 x - \sin^2 x$

Solve the following trigonometric equations on the interval $\theta = [-2\pi, 2\pi]$ exactly.

322. $6\cos^2 x - 3 = 0$

323. $\sec^2 x - 2\sec x + 1 = 0$

Solve the following logarithmic equations.

324. $5^x = 16$

325. $\log_2(x + 4) = 3$

Are the following functions one-to-one over their domain of existence? Does the function have an inverse? If so, find the inverse $f^{-1}(x)$ of the function. Justify your answer.

326. $f(x) = x^2 + 2x + 1$

327. $f(x) = \frac{1}{x}$

For the following problems, determine the largest domain on which the function is one-to-one and find the inverse on that domain.

328. $f(x) = \sqrt{9 - x}$

329. $f(x) = x^2 + 3x + 4$

330. A car is racing along a circular track with diameter of 1 mi. A trainer standing in the center of the circle marks his progress every 5 sec. After 5 sec, the trainer has to turn 55° to keep up with the car. How fast is the car traveling?

For the following problems, consider a restaurant owner who wants to sell T-shirts advertising his brand. He recalls that there is a fixed cost and variable cost, although he does not remember the values. He does know that the T-shirt printing company charges $440 for 20 shirts and $1000 for 100 shirts.

331. a. Find the equation $C = f(x)$ that describes the total cost as a function of number of shirts and b. determine how many shirts he must sell to break even if he sells the shirts for $10 each.

332. a. Find the inverse function $x = f^{-1}(C)$ and describe the meaning of this function. b. Determine how many shirts the owner can buy if he has $8000 to spend.

For the following problems, consider the population of Ocean City, New Jersey, which is cyclical by season.

333. The population can be modeled by $P(t) = 82.5 - 67.5\cos[(\pi/6)t]$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). During a year, in what intervals is the population less than 20,000? During what intervals is the population more than 140,000?

334. In reality, the overall population is most likely increasing or decreasing throughout each year. Let's reformulate the model as $P(t) = 82.5 - 67.5\cos[(\pi/6)t] + t$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). When is the first time the population reaches 200,000?

For the following problems, consider radioactive dating. A human skeleton is found in an archeological dig. Carbon dating is implemented to determine how old the skeleton is by using the equation $y = e^{rt}$, where y is the percentage of radiocarbon still present in the material, t is the number of years passed, and $r = -0.0001210$ is the decay rate of radiocarbon.

335. If the skeleton is expected to be 2000 years old, what percentage of radiocarbon should be present?

336. Find the inverse of the carbon-dating equation. What does it mean? If there is 25% radiocarbon, how old is the skeleton?